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The Wave, The Cylinder, and The Plate

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§ Abstract

In this report, a simple fluid–structure interaction problem is proposed as a benchmark test for fluid–structure codes. The report details the solution to the problem of a plane wave scattering off an elastic cylinder with internal structure in an acoustic medium. The particular system analyzed is an infinitely long cylinder with an internal hinged plate, and as such, the problem is two dimensional in nature. The analysis sets up the basics of acoustic scattering from elastic structures and then proceeds to develop the particular solution for the chosen structure. The complete solution to the steady state problem is developed in terms of harmonic series in the polar angle. The final expressions are in terms of infinite series over the cylindrical shell harmonics and permit comparisons and validation of results from numerical techniques such as finite difference and finite element methods. It is noted that the geometry, although simple, embodies some of the main features of more complicated problems and thus makes the given problem suitable for use in benchmarking and verification.

§1. Introduction

The general problem of fluid–structure interaction is a very complicated coupled initial boundary value problem. Analytic techniques, though able to provide insight and/or full solutions to many particular problems, are often incapable of accurately handling irregular structural features. One proposal for handling more general fluid–structure interaction problems is the use of finite element or finite difference methods. As a means of benchmarking and validating such procedures, a simple test problem is proposed and solved analytically in this report. The chosen problem is designed to be geometrically simple yet complex enough to embody some of the main coupling features that a fluid–structure code would have to capture.

The system analyzed consists of a plane wave striking an infinite cylinder submerged in an infinite acoustic medium. The cylinder is of radius R_s with density ρ_s and thickness h_s. There is an internal hinged plate at angle θ_p. The plate has thickness h_p and density ρ_p. The material response of the plate and the shell is elastic with elastic constants (E_p, ν_p) and (E_s, ν_s) respectively. After the plane wave strikes the cylinder it causes scattered waves to be generated. The purpose here is to calculate the scattered waves. See Figure **1.1**.

To perform the calculation, the problem is broken up into several smaller problems. First the problem is broken up into the problem of an incident plane wave scattering off a rigid target. Then a particular elastic target problem is solved and combined with this rigid target result to generate the scattered field from an elastic target under the influence of a plane wave. The setup of the scattering problem follows directly from the monograph of JUNGER AND FEIT [1972]. In what follows, this work is cited for many results and much material is repeated to make the presentation as self contained as is reasonably possible. The method of Lagrange multipliers is used to generate the normal modes of vibration of the shell–plate structure; the elasticity analysis closely mirrors the work of BJARNASON, ACHENBACH, AND IGUSA [1992].

§2. Pressure Decomposition

The goal is to develop the expression for the pressure field in the acoustic medium as a function of position and time. Because of the linearity of the systems involved, the problem may be decomposed into the sum of several simpler problems. First, the total pressure response is written as

$$p = p^i + p^{se} \,, \tag{2.1}$$

where p^i is the incident plane wave pressure field and p^{se} is the scattered pressure field. The scattered field is further decomposed into two parts:

$$p^{se} = p^{s\infty} + p^r \,, \tag{2.2}$$

where $p^{s\infty}$ is the scattered pressure field from an equivalent rigid target and p^r is the contribution to the scattered pressure field due to the target's elasticity which is sometimes termed the radiated field.

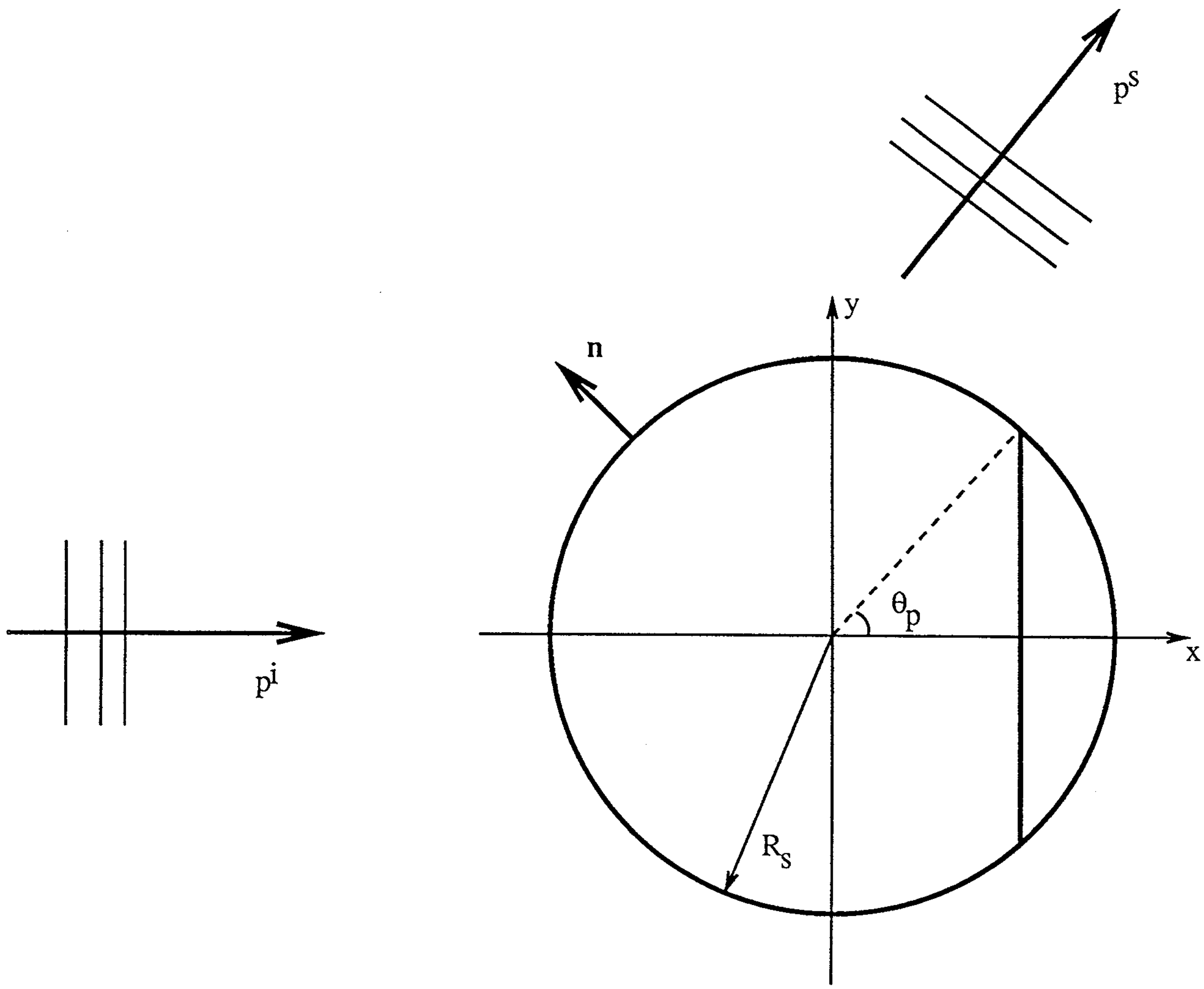

FIGURE 1.1. Problem Geometry

§3. Rigid Scatter

Begin by considering the case of a rigid cylinder. Decompose the total pressure field into two parts: the incident pressure and the scattered pressure; i.e.

$$\widehat{p} = p^i + p^{s\infty}\,, \tag{3.1}$$

where the superscript ∞ is to remind the reader that this is the scattered pressure field when the cylinder has an infinite impedance. Based on the geometry shown in Figure **1.1**, the incident pressure field may be written as

$$p^i = P^i \exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\,, \tag{3.2}$$

where P^i is the magnitude of the incident wave, i is the imaginary unit, $\boldsymbol{r}$ is the position vector, ω is the frequency of the incoming wave, t is the time, and the wave vector $\boldsymbol{k} = k\boldsymbol{e}_x$ where $\boldsymbol{e}_x$ is the unit vector in the x–direction.

Remark 3.1.
The linearity of the governing equations involved affords us the convenience of using complex number notation. By using the complex number representation to the incident pressure field, two boundary value problems end up being solved. If the solution to the complex input p^i is denoted by ζ, then $\mathrm{Re}\{\zeta\}$ is the solution to the input $\mathrm{Re}\{p^i\}$ and $\mathrm{Im}\{\zeta\}$ is the solution to the input $\mathrm{Im}\{p^i\}$. $\square$

Remark 3.2.
In what follows the always present $\exp[-i\omega t]$ will be omitted for notational clarity. In cases where the inclusion of this term will aid in the clarity of the presentation, it will be utilized. $\square$

All the quantities in the incident pressure field are known. To calculate the rigid scatter pressure field, the momentum balance equation must be solved. This equation is given by

$$\nabla p = -\rho_f \ddot{\boldsymbol{u}}\,, \tag{3.3}$$

where ρ_f is the density of the fluid and $\boldsymbol{u}$ is the fluid displacement. Because the cylinder is assumed to be rigid

$$\widehat{u}_r = \widehat{\boldsymbol{u}}(R_s) \cdot \boldsymbol{n} \equiv 0\,, \tag{3.4}$$

where $\boldsymbol{n}$ is the cylinder normal. Therefore, the boundary condition for the scattered field is given by

$$u_r^{s\infty}(R_s) = -u_r^i(R_s) \tag{3.5}$$

and likewise for the accelerations. Utilizing the momentum balance equation this may be written as:

$$\rho_f \ddot{u}_r^{s\infty} = [\nabla p^i \cdot \boldsymbol{n}] \ \text{ at } r = R_s\,. \tag{3.6}$$

Consider now the expression for the incident pressure field. Based on the given geometry

$$p^i = P^i \exp[ikr\cos\theta] = P^i \sum_{n=0}^{\infty} \bar{\epsilon}_n i^n J_n(kr)\cos(n\theta)\,, \tag{3.7}$$

where

$$\bar{\epsilon}_n = \begin{cases} 1, & \text{if } n = 0 \\ 2 & \text{if } n > 0, \end{cases} \tag{3.8}$$

and $J_n(\cdot)$ is the Bessel function of the first kind of order n. The summation expression above can be deduced from Equations 9.1.44 and 9.1.45 in Abramowitz and Stegun [1972] and Euler's exponential formula. Equation (3.7) may now be inserted into Equation (3.6) to give

$$\begin{aligned} \ddot{u}_r^{s\infty}(R_s,\theta) &= \nabla \left[\frac{P^i}{\rho_f} \sum_{n=0}^{\infty} \bar{\epsilon}_n i^n J_n(kr)\cos(n\theta) \right] \cdot \boldsymbol{e}_r \Bigg|_{r=R_s} \\ &= \frac{k}{\rho_f} P^i \sum_{n=0}^{\infty} \bar{\epsilon}_n i^n J_n'(kR_s)\cos(n\theta)\,. \end{aligned} \tag{3.9}$$

From the above, the Fourier coefficients to the boundary condition can be directly observed to be

$$(\ddot{U}_r^{s\infty})_n = \frac{k}{\rho_f} P^i \bar{\epsilon}_n i^n J_n'(kR_s)\,. \tag{3.10}$$

Once the Fourier coefficients to the boundary condition are known, the solution to the generated pressure field may be computed from Equation 8.41 in JUNGER AND FEIT[1972]. In the present context, this equation reduces to

$$\begin{aligned} p^{s\infty} &= -\rho_f \sum_{n=0}^{\infty} (\ddot{U}_r^{s\infty})_n \frac{H_n(kr)}{kH_n'(kR_s)} \cos(n\theta) \\ &= -P^i \sum_{n=0}^{\infty} \bar{\epsilon}_n i^n \frac{J_n'(kR_s) H_n(kr)}{H_n'(kR_s)} \cos(n\theta)\,, \end{aligned} \tag{3.11}$$

where $H_n(\cdot)$ is the Hankel function of the first kind of order n.

It remains now to calculate the expression for the elastically radiated contribution to the scattered pressure field. This is considered in the next section.

§4. Elastic Scattering

To solve for the elastically radiated contribution to the scattered field, the boundary condition at the interface must first be deduced. From the balance equation (3.3) applied at the cylinder surface

$$\frac{\partial p}{\partial n}(R_s, \theta) = -\rho_f \ddot{u}_r(R_s, \theta)\,. \tag{4.1}$$

Recall that the pressure has been decomposed into three parts and that the normal gradient of $p^i + p^{s\infty}$ is identically zero on the cylinder surface by construction; see Equation (3.4). Therefore, the boundary condition of interest is given by

$$\frac{\partial p^r}{\partial n}(R_s, \theta) = -\rho_f \ddot{u}_r(R_s, \theta)\,, \tag{4.2}$$

where $u_r(R_s, \theta)$ is the radial motion of the elastic structure under the total pressure loading $p = p^i + p^{s\infty} + p^r$. To solve this coupled problem, the following decompositions prove useful. First, express the radial structure motion in a Fourier series in θ; ie.

$$u_r(R_s, \theta) = \sum_{n=0}^{\infty} U_n \cos(n\theta)\,. \tag{4.3}$$

Then it is possible to write the radiated pressure due to this motion in the form

$$p^r = \sum_{n=0}^{\infty} U_n z_n^a \cos(n\theta)\,, \tag{4.4}$$

where z_n^a can be termed an acoustic impedance which is given by the following expression

$$z_n^a = \omega^2 \rho_f \frac{H_n(kr)}{kH_n'(kR_s)} . \tag{4.5}$$

Equation (4.5) follows directly from JUNGER AND FEIT [1972] Equation 8.41 and will not be elaborated on in this report.

In the simple case of an empty elastic cylinder, it is possible to write the cylinder response as

$$u_r = \sum_{n=0}^{\infty} -p_n \cos(n\theta)/Z_n^s , \tag{4.6}$$

where $p_n = p_n^i + p_n^{s\infty} + p_n^r$ and Z_n^s is known as the structural impedance. The forms for the Fourier coefficients p_n^i and $p_n^{s\infty}$ are readily deducible from Equations (3.7) and (3.11). Combining Equations (4.3),(4.4), and (4.6) it is possible to solve for the Fourier coefficients of the cylinder displacement as

$$U_n = -\left.\frac{p_n^i + p_n^{s\infty}}{Z_n^s + z_n^a}\right|_{r=R_s} . \tag{4.7}$$

The radiated pressure field due to the cylinder's elasticity can be computed by plugging Equation (4.7) into Equation (4.4).

The specific form of Z_n^s is omitted here because this simple situation where the modes of the structural response uncouple as in Equation (4.6), does not hold in the presence of the internal plate. Whenever the cylinder contains internal structure, all the modes of the system are coupled together. In the case studied here, one finds (as will be shown in §5) that the form of the response is given by

$$U_n = L_n[p] , \tag{4.8}$$

where

$$L_n[\cdot] = E_n[\cdot]_n + F_n \sum_{m=0}^{\infty} A_m[\cdot]_m + G_n \sum_{m=0}^{\infty} B_m[\cdot]_m . \tag{4.9}$$

In the above, $(E_n, F_n, G_n, A_m, B_m)$ are all unknowns that will be calculated later from the consideration of particular shell and plate theories. From Equations (4.8) and (4.9), all the modes of the response are seen to be coupled; ie. the n-th mode of the pressure loading is seen to affect all the modes of the displacement response. To solve this more difficult set of equations, first note Equations (3.1) and (4.4) imply that

$$p_n = \widehat{p}_n + z_n^a U_n , \tag{4.10}$$

where it is understood that all quantities are evaluated at $r = R_S$ for the remainder of this section. Insert Equation (4.10) into Equation (4.8) to yield

$$U_n = L_n[\widehat{p}] + L_n[z^a U] . \tag{4.11}$$

This equation is solved for U_n in roughly the same way one solves the integral equation

$$f(x) = g(x) + h(x) \int f(y)\, dy \tag{4.12}$$

for $f(x)$; ie. integrate both sides and solve for $\int f(y)$ and plug back in to get $f(x)$. In the present situation the algebra is somewhat more complicated owing to the presence of two sums as opposed to the one represented by the integral.

To begin, separate the free U_n's in Equation (4.11) and rewrite as

$$U_n = \frac{1}{1 - E_n z_n^a} \left[L_n[\hat{p}] + F_n \left(\sum_{m=0}^{\infty} A_m z_m^a U_m \right) + G_n \left(\sum_{m=0}^{\infty} B_m z_m^a U_m \right) \right] \Bigg|_{r=R_s} . \tag{4.13}$$

The task now is to compute the unknown sums that are prefixed by F_n and G_n. This is accomplished by generating two linear equations in the two unknown sums. The two equations are formed by multiplying Equation (4.13) by $z_n^a A_n$ and summing over n and by multiplying Equation (4.13) by $z_n^a B_n$ and summing over n. Doing so, and rewriting in matrix form yields the following system of equations for the unknown sums:

$$\begin{bmatrix} 1 - \sum_{n=0}^{\infty} \frac{A_n z_n^a F_n}{1 - E_n z_n^a} & -\sum_{n=0}^{\infty} \frac{A_n z_n^a G_n}{1 - E_n z_n^a} \\ -\sum_{n=0}^{\infty} \frac{B_n z_n^a F_n}{1 - E_n z_n^a} & 1 - \sum_{n=0}^{\infty} \frac{B_n z_n^a G_n}{1 - E_n z_n^a} \end{bmatrix} \begin{pmatrix} \sum_{m=0}^{\infty} A_m z_m^a U_m \\ \sum_{m=0}^{\infty} B_m z_m^a U_m \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{A_n z_n^a L_n[\hat{p}]}{1 - E_n z_n^a} \\ \sum_{n=0}^{\infty} \frac{B_n z_n^a L_n[\hat{p}]}{1 - E_n z_n^a} \end{pmatrix} . \tag{4.14}$$

Equation (4.14) can now be solved for $(\sum_{m=0}^{\infty} A_m z_m^a U_m)$ and $(\sum_{m=0}^{\infty} B_m z_m^a U_m)$ to give

$$\sum_{m=0}^{\infty} A_m z_m^a U_m = \frac{\left(\sum_{n=0}^{\infty} \frac{A_n z_n^a L_n[\hat{p}]}{1 - E_n z_n^a} \right) \left(1 - \sum_{n=0}^{\infty} \frac{B_n z_n^a G_n}{1 - E_n z_n^a} \right) + \left(\sum_{n=0}^{\infty} \frac{B_n z_n^a L_n[\hat{p}]}{1 - E_n z_n^a} \right) \left(\sum_{n=0}^{\infty} \frac{A_n z_n^a G_n}{1 - E_n z_n^a} \right)}{\left(1 - \sum_{n=0}^{\infty} \frac{A_n z_n^a F_n}{1 - E_n z_n^a} \right) \left(1 - \sum_{n=0}^{\infty} \frac{B_n z_n^a G_n}{1 - E_n z_n^a} \right) - \left(\sum_{n=0}^{\infty} \frac{A_n z_n^a G_n}{1 - E_n z_n^a} \right) \left(\sum_{n=0}^{\infty} \frac{B_n z_n^a F_n}{1 - E_n z_n^a} \right)} \tag{4.15}$$

and

$$\sum_{m=0}^{\infty} B_m z_m^a U_m = \frac{\left(\sum_{n=0}^{\infty} \frac{B_n z_n^a L_n[\hat{p}]}{1 - E_n z_n^a} \right) \left(1 - \sum_{n=0}^{\infty} \frac{A_n z_n^a F_n}{1 - E_n z_n^a} \right) + \left(\sum_{n=0}^{\infty} \frac{A_n z_n^a L_n[\hat{p}]}{1 - E_n z_n^a} \right) \left(\sum_{n=0}^{\infty} \frac{B_n z_n^a F_n}{1 - E_n z_n^a} \right)}{\left(1 - \sum_{n=0}^{\infty} \frac{A_n z_n^a F_n}{1 - E_n z_n^a} \right) \left(1 - \sum_{n=0}^{\infty} \frac{B_n z_n^a G_n}{1 - E_n z_n^a} \right) - \left(\sum_{n=0}^{\infty} \frac{A_n z_n^a G_n}{1 - E_n z_n^a} \right) \left(\sum_{n=0}^{\infty} \frac{B_n z_n^a F_n}{1 - E_n z_n^a} \right)} . \tag{4.16}$$

Equations (4.15) and (4.16) can now be plugged back into Equation (4.13) to get the final expression for U_n. Once done, this expression can be inserted into the impedance expression, Equation (4.4), to get the radiated pressure field due to the structure's elasticity.

It remains to calculate the expressions for $(E_n, F_n, G_n, A_m, B_m)$. This will be done in the next section.

§5. Elastic Structural Response

To calculate the unknown coefficients in Equation (4.9), the elastic response of the shell–plate structure to an imposed pressure loading must be calculated.

Because most underwater structures of interest can easily be characterized by thickness to radius ratios of much less than one, Donnell's equations of motion for the shell are chosen. For a discussion of the basic assumptions involved in this shell theory see JUNGER AND FEIT [Chap 9, 1972], KRAUS [Chap 6, 1967], or DONNELL [1933]. From the symmetry of the loading and the chosen geometry, plane strain conditions may be assumed. Hence, the axial displacements $u_z \equiv 0$ and $\frac{\partial}{\partial z}(\cdot) \equiv 0$. Therefore, there are only two non-trivial momentum balance equations:

$$\frac{1}{R_s^2}\left(\frac{\partial^2 u_\theta}{\partial\theta^2} + \frac{\partial u_r}{\partial\theta}\right) - \frac{1}{c_s^2}\ddot{u}_\theta = 0 \tag{5.1}$$

and

$$\frac{1}{R_s^2}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right) + \frac{\beta^2}{R_s^2}\frac{\partial^4 u_r}{\partial\theta^4} + \frac{1}{c_s^2}\ddot{u}_r + p\frac{1-\nu_s^2}{E_s h_s} = 0\,, \tag{5.2}$$

where the imposed pressure loading, p, is taken as positive inward, $\beta^2 = h_s^2/(12R_s^2)$, and $c_s^2 = E_s/[(1-\nu_s^2)\rho_s]$.

The motion of the plate is characterized by the displacement field $\boldsymbol{w}$ which is only a function of y by the plane strain assumption employed for the shell. Therefore, there are two non-trivial momentum balance equations for the plate. When higher order coupling terms between the longitudinal and transverse motion are ignored, the transverse balance of momentum (see e.g. SZILARD [4.2.12, 1974]) becomes:

$$\frac{E_p h_p^3}{12(1-\nu^2)}\frac{\partial^4 w_x}{\partial y^4} + \rho_p h_p \ddot{w}_x = 0\,. \tag{5.3}$$

For the longitudinal balance equation, one can apply energy considerations as partially outlined in TIMOSHENKO AND WOINOWSKY-KRIEGER [Art. 92, 1959] to yield

$$\frac{E_p h_p}{1-\nu^2}\frac{\partial^2 w_y}{\partial y^2} - \rho_p h_p \ddot{w}_y = 0\,. \tag{5.4}$$

Equations (5.1)–(5.4) are coupled together through the hinge constraint that says that the shell and the plate must have the same displacement at their intersection points. In terms of the present notation, this is expressed as

$$w_x \boldsymbol{e}_x + w_y \boldsymbol{e}_y = u_r \boldsymbol{e}_r + u_\theta \boldsymbol{e}_\theta \quad \text{at } r = R_s,\ \theta = \pm\theta_p\,. \tag{5.5}$$

The solution of Equations (5.1)–(5.4), given the constraint (5.5), is mostly easily effected by using minimization techniques with Lagrange multipliers. To this end, the energy forms per unit depth that generate the above given balance equations are summarized below. The kinetic energy of the shell is given by:

$$T_{\text{shell}} = \frac{1}{2}\rho_s h_s \int_0^{2\pi} (\dot{u}_\theta^2 + \dot{u}_r^2) R_s \, d\theta \,. \tag{5.6}$$

The potential energy of the shell is given by:

$$V_{\text{shell}} = \frac{1}{2}\frac{E_s h_s}{1-\nu_s^2} \int_0^{2\pi} \left[\frac{1}{R_s^2}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right)^2 + \frac{\beta^2}{R_s^2}\left(\frac{\partial^2 u_r}{\partial \theta^2}\right)^2 \right] R_s \, d\theta \,. \tag{5.7}$$

And the loading energy on the shell is given by

$$U_{\text{shell}} = -\int_0^{2\pi} p(\theta) u_r(\theta) R_s \, d\theta \,, \tag{5.8}$$

where the minus sign appears because the loading is taken as positive inwards. The plate kinetic energy is given by

$$T_{\text{plate}} = \frac{1}{2}\rho_p h_p \int_{-L/2}^{L/2} [\dot{w}_x^2 + \dot{w}_y^2] \, dy \,, \tag{5.9}$$

where $L = 2R_s \sin\theta_p$. And the potential energy of the plate is given by

$$V_{\text{plate}} = \frac{1}{2}\frac{E_p h_p}{1-\nu_p^2} \int_{-L/2}^{L/2} \left[\left(\frac{\partial w_y}{\partial y}\right)^2 + \frac{h_p^2}{12}\left(\frac{\partial^2 w_x}{\partial y^2}\right)^2 \right] dy \,. \tag{5.10}$$

The fact that these are the correct expressions for the kinetic, potential, and loading energies can be easily verified by first forming the Lagrangian over a specified time interval as

$$\widehat{\mathsf{L}}(u_r, u_\theta, w_x, w_y) = \int_{t_0}^{t_1} (T_{\text{shell}} + T_{\text{plate}} - V_{\text{shell}} - V_{\text{plate}} + U_{\text{shell}}) \, dt \,, \tag{5.11}$$

and then by taking the variational derivatives of $\widehat{\mathsf{L}}$ with respect to each of its four arguments. [Note that all the variations are zero at t_0 and t_1 (ie. the temporal endpoints are multiples of $2\pi/\omega$, since the analysis is restricted to time harmonic solutions), the shell displacement variations are 2π periodic, and the plate displacement variations are zero at $y = \pm L/2$.]

To incorporate constraint (5.5), it is broken up into two independent scalar constraints

$$\begin{aligned} \Psi_1 &= w_x(R_s \sin\theta_p) - u_r(\theta_p)\cos\theta_p + u_\theta(\theta_p)\sin\theta_p \equiv 0 \\ \Psi_2 &= w_y(R_s \sin\theta_p) - u_r(\theta_p)\sin\theta_p - u_\theta(\theta_p)\cos\theta_p \equiv 0 \,. \end{aligned} \tag{5.12}$$

The constraints are then appended to the previous Lagrangian to give the full system Lagrangian as

$$\mathsf{L}(u_r, u_\theta, w_x, w_y, \lambda_1, \lambda_2) = \widehat{\mathsf{L}} + \lambda_1\Psi_1 + \lambda_2\Psi_2 \,. \tag{5.13}$$

At this juncture, it proves useful to make use of the system's symmetry and perform a spatially harmonic decomposition of the response. This yields

$$
\begin{aligned}
u_r &= \sum_{n=0}^{\infty} U_n \cos n\theta e^{-i\omega t}\,, \\
u_\theta &= \sum_{n=0}^{\infty} V_n \sin n\theta e^{-i\omega t}\,, \\
w_x &= c e^{-i\omega t} + \sum_{j=1}^{\infty} b_j \cos(2j-1)\frac{\pi y}{L} e^{-i\omega t}\,, \\
w_y &= \sum_{j=1}^{\infty} a_j \sin(2j-1)\frac{\pi y}{L} e^{-i\omega t}\,,
\end{aligned}
\tag{5.14}
$$

where $V_0 \equiv 0$ and c is the rigid body displacement of the plate due to the symmetric shell displacement. Note that the Lagrange multipliers are also given by expressions of the form

$$\lambda_j = \lambda_j e^{-i\omega t} \quad j \in (1,2)\,. \tag{5.15}$$

The harmonic coefficients (U_n, V_n, b_j, a_j), the rigid body motion of the plate c, and the Lagrange multipliers are all unknowns. To solve for them, the assumed forms (5.14) and (5.15) are inserted into the Lagrangian (5.13) and the spatial and temporal integrations are carried out. Then the variations are taken with respect to $(U_n, V_n, b_j, a_j, c, \lambda_1, \lambda_2)$ to generate a set of governing algebraic equations. In performing the above manipulations one must carefully take into account the fact that only the real part (or the imaginary part) of a quantity should be used in calculating energy like terms; ie. terms of the generic form

$$\text{generic term} = uv \quad u, v \in \mathbb{C} \tag{5.16}$$

must be written out for the real case as

$$\text{generic term} = \left[\frac{1}{2}(u+\bar{u})\frac{1}{2}(v+\bar{v})\right] \tag{5.17}$$

where the superposed bar indicates the complex conjugate of a quantity.

If the above manipulations are carried out, the energy expressions become:

$$T_{\text{shell}} = \frac{1}{2} R_s \rho_s h_s \omega^2 \pi \sum_{n=0}^{\infty} \epsilon_n (U_n^2 + V_n^2)\,, \tag{5.18}$$

$$V_{\text{shell}} = \frac{1}{2}\frac{E_s h_s \pi}{(1-\nu_s^2)R_s} \sum_{n=0}^{\infty} \epsilon_n \left[(nV_n + U_n)^2 + \beta^2 n^4 U_n^2\right]\,, \tag{5.19}$$

$$U_{\text{shell}} = -R_s \pi \epsilon_n p_n U_n\,, \tag{5.20}$$

$$T_{\text{plate}} = \frac{1}{4}\rho_p h_p L \omega^2 \left[2c^2 + \sum_{j=1}^{\infty} a_j^2 + b_j^2 + 2cb_j\mu_j\right]\,, \tag{5.21}$$

$$V_{\text{plate}} = \frac{1}{4}\rho_p h_p L \sum_{j=1}^{\infty} a_j^2 \omega_{(L)j}^2 + b_j^2 \omega_{(T)j}^2\,. \tag{5.22}$$

In the above,

$$\mu_j = \frac{2}{L}\int_{-L/2}^{L/2} \cos(2j-1)\frac{\pi y}{L}\,dy = (-1)^j \frac{4}{(2j-1)\pi}\,, \tag{5.23}$$

and

$$\epsilon_n = \begin{cases} 2, & \text{if } n = 0 \\ 1 & \text{if } n > 0. \end{cases} \tag{5.24}$$

The symbol $\omega_{(L)j}$ denotes the in vacuo longitudinal plate frequencies and is given by:

$$\omega^2_{(L)j} = \frac{(2j-1)^2\pi^2}{L^2}\frac{E_p}{(1-\nu_p^2)\rho_p}\,. \tag{5.25}$$

The in vacuo transverse plate frequencies $\omega_{(T)j}$ are defined by:

$$\omega^2_{(T)j} = \frac{(2j-1)^4\pi^4}{L^4}\frac{E_p h_p^2}{12(1-\nu_p^2)\rho_p}\,. \tag{5.26}$$

Using the same substitutions the constraints become

$$\begin{aligned} \Psi_1 &= c - \sum_{n=0}^{\infty} U_n \cos n\theta_p \cos\theta_p + \sum_{n=0}^{\infty} V_n \sin n\theta_p \sin\theta_p \equiv 0 \\ \Psi_2 &= \sum_{j=1}^{\infty}(-1)^{j-1}a_j - \sum_{n=0}^{\infty} U_n \cos n\theta_p \sin\theta_p - \sum_{n=0}^{\infty} V_n \sin n\theta_p \cos\theta_p \equiv 0\,. \end{aligned} \tag{5.27}$$

In the preceding equations the time dependence has been suppressed. When included, the time dependence yields an identical multiplicative constant to all the equations and hence will drop out of the formulation. Therefore, for clarity, it has been omitted.

Equations (5.18)–(5.27) may now be inserted into Equation (5.13) to give the harmonically decomposed form of the system Lagrangian. The set of algebraic equations that must be solved for the unknowns $(U_n, V_n, b_j, a_j, c, \lambda_1, \lambda_2)$ is made up of the equations defining the critical point of the Lagrangian. Taking first derivatives of the Lagrangian with respect to each unknown gives:

$$\begin{aligned} R_s\rho_s h_s\omega^2\pi\epsilon_n U_n &- \frac{E_s h_s \pi}{(1-\nu_s^2)R_s}\epsilon_n[(nV_n + U_n) + \beta^2 n^4 U_n] - R_s\pi\epsilon_n p_n \\ &- \lambda_1 \cos n\theta_p \cos\theta_p - \lambda_2 \cos n\theta_p \sin\theta_p = 0\,, \end{aligned} \tag{5.28}$$

$$\begin{aligned} R_s\rho_s h_s\omega^2\pi\epsilon_n V_n &- \frac{E_s h_s \pi}{(1-\nu_s^2)R_s}\epsilon_n(n^2 V_n + nU_n) \\ &+ \lambda_1 \sin n\theta_p \sin\theta_p - \lambda_2 \sin n\theta_p \cos\theta_p = 0\,, \end{aligned} \tag{5.29}$$

$$\frac{1}{2}\rho_p h_p L\omega^2 a_j - \frac{1}{2}\rho_p h_p L\omega^2_{(L)j}a_j + (-1)^{j-1}\lambda_2 = 0\,, \tag{5.30}$$

$$\frac{1}{2}\rho_p h_p L\omega^2(b_j + c\mu_j) - \frac{1}{2}\rho_p h_p L\omega^2_{(T)j} b_j = 0\,, \tag{5.31}$$

$$\frac{1}{2}\rho_p h_p L\omega^2 \left[2c + \sum_{j=1}^{\infty} b_j\mu_j\right] + \lambda_1 = 0\,, \tag{5.32}$$

$$\Psi_1 = 0\,, \tag{5.33}$$

and

$$\Psi_2 = 0\,. \tag{5.34}$$

Equations (5.28)–(5.34) represents an inhomogeneous system of $(4n+3,\ n \to \infty)$ algebraic equations in $(4n+3,\ n \to \infty)$ unknowns. The goal is to solve for U_n in terms of p; so that the developments of §4 may be applied. It helps to first nondimensionalize the governing equations by introducing the following quantities:

$$\Omega = \omega\frac{R_s}{c_s}, \quad \Omega_{(L)j} = \omega_{(L)j}\frac{R_s}{c_s}, \quad \Omega_{(T)j} = \omega_{(T)j}\frac{R_s}{c_s}\,, \tag{5.35}$$

$$\tilde{p}_n = p_n\frac{R_s}{h_s\rho_s c_s^2}, \quad \tilde{\lambda}_j = \lambda_j\frac{1}{h_s\rho_s c_s^2\pi}, \quad M = 2\sin\theta_p\frac{\rho_p h_p L}{\rho_s h_s 2\pi R_s}\,, \tag{5.36}$$

$$\tilde{U}_n = \frac{U_n}{R_s}, \quad \tilde{V}_n = \frac{V_n}{R_s}\,, \tag{5.37}$$

and

$$\tilde{a}_j = \frac{a_j}{L}, \quad \tilde{b}_j = \frac{b_j}{L}, \quad \tilde{c} = \frac{c}{L}\,. \tag{5.38}$$

Utilizing these nondimensional quantities Equations (5.28)–(5.34) appear in the simpler form:

$$(\Omega^2 - \beta^2 n^4 - 1)\epsilon_n\tilde{U}_n - n\epsilon_n\tilde{V}_n - \epsilon_n\tilde{p}_n - \tilde{\lambda}_1\cos n\theta_p\cos\theta_p - \tilde{\lambda}_2\cos n\theta_p\sin\theta_p = 0\,, \tag{5.39}$$

$$(\Omega^2 - n^2)\epsilon_n\tilde{V}_n - n\epsilon_n\tilde{U}_n + \tilde{\lambda}_1\sin n\theta_p\sin\theta_p - \tilde{\lambda}_2\sin n\theta_p\cos\theta_p = 0\,, \tag{5.40}$$

$$M(\Omega^2 - \Omega^2_{(L)j})\tilde{a}_j + (-1)^{j-1}\tilde{\lambda}_2 = 0\,, \tag{5.41}$$

$$(\Omega^2 - \Omega^2_{(T)j})\tilde{b}_j + \Omega^2\tilde{c}\mu_j = 0\,, \tag{5.42}$$

$$M\Omega^2\left[2\tilde{c} + \sum_{j=1}^{\infty}\tilde{b}_j\mu_j\right] + \tilde{\lambda}_1 = 0\,, \tag{5.43}$$

$$\tilde{\Psi}_1 = 2\sin\theta_p\tilde{c} - \sum_{n=0}^{\infty}\tilde{U}_n\cos n\theta_p\cos\theta_p + \sum_{n=0}^{\infty}\tilde{V}_n\sin n\theta_p\sin\theta_p = 0\,, \tag{5.44}$$

and

$$\tilde{\Psi}_2 = \sum_{j=1}^{\infty}(-1)^{j-1}2\sin\theta_p\tilde{a}_j - \sum_{n=0}^{\infty}\tilde{U}_n\cos n\theta_p\sin\theta_p - \sum_{n=0}^{\infty}\tilde{V}_n\sin n\theta_p\cos\theta_p = 0\,. \tag{5.45}$$

First, Equations (5.39) and (5.40) are solved for $\tilde{U}_n$ and $\tilde{V}_n$ in terms of the pressure and the Lagrange multipliers. This yields:

$$\begin{aligned} \tilde{U}_n = \frac{(\Omega^2 - n^2)}{D_n(\Omega)}\tilde{p}_n &+ \frac{(\Omega^2 - n^2)\cos n\theta_p \cos\theta_p - n\cos n\theta_p \sin\theta_p}{\epsilon_n D_n(\Omega)}\tilde{\lambda}_1 \\ &+ \frac{(\Omega^2 - n^2)\sin n\theta_p \sin\theta_p + n\sin n\theta_p \cos\theta_p}{\epsilon_n D_n(\Omega)}\tilde{\lambda}_2 \end{aligned} \tag{5.46}$$

and

$$\begin{aligned} \tilde{V}_n = \frac{n}{D_n(\Omega)}\tilde{p}_n &+ \frac{n\cos n\theta_p \cos\theta_p - (\Omega^2 - \beta^2 n^4 - 1)\cos n\theta_p \sin\theta_p}{\epsilon_n D_n(\Omega)}\tilde{\lambda}_1 \\ &+ \frac{n\sin n\theta_p \sin\theta_p + (\Omega^2 - \beta^2 n^4 - 1)\sin n\theta_p \cos\theta_p}{\epsilon_n D_n(\Omega)}\tilde{\lambda}_2 , \end{aligned} \tag{5.47}$$

where the quantity $D_n(\Omega)$ is the characteristic equation of the shell, and is given by:

$$D_n(\Omega) = \Omega^4 - (\beta^2 n^4 + n^2 + 1)\Omega^2 + \beta^2 n^6 . \tag{5.48}$$

Next, Equation (5.41) is solved for $\tilde{a}_j$ in terms of $\tilde{\lambda}_2$ as:

$$\tilde{a}_j = \frac{(-1)^j}{M(\Omega^2 - \Omega^2_{(L)j})}\tilde{\lambda}_2 , \tag{5.49}$$

and Equations (5.42) and (5.43) are solved for $\tilde{c}$ in terms of $\tilde{\lambda}_1$ as:

$$\tilde{c} = -\frac{\tilde{\lambda}_1}{M\Omega^2\left[2 - \sum_{j=1}^{\infty}\frac{\Omega^2}{\Omega^2 - \Omega^2_{(T)j}}\mu_j^2\right]} . \tag{5.50}$$

Equations (5.46)–(5.50) may now be inserted into the constraint Equations (5.44) and (5.45) to yield a system of equations for the Lagrange multipliers. Written in matrix form, one has

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\begin{pmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} , \tag{5.51}$$

where the coefficients are given by:

$$\begin{aligned} N_{11} = &\frac{-2\sin\theta_p}{M\Omega^2\left[2 - \sum_{j=1}^{\infty}\frac{\Omega^2}{\Omega^2 - \Omega^2_{(T)j}}\mu_j^2\right]} \\ &- \sum_{n=0}^{\infty}\cos n\theta_p \cos\theta_p \frac{(\Omega^2 - n^2)\cos n\theta_p \cos\theta_p - n\cos n\theta_p \sin\theta_p}{\epsilon_n D_n(\Omega)} \\ &+ \sum_{n=0}^{\infty}\sin n\theta_p \sin\theta_p \frac{n\cos n\theta_p \cos\theta_p - (\Omega^2 - \beta^2 n^4 - 1)\cos n\theta_p \sin\theta_p}{\epsilon_n D_n(\Omega)} , \end{aligned} \tag{5.52}$$

$$\begin{aligned} N_{12} = &-\sum_{n=0}^{\infty} \cos n\theta_p \cos\theta_p \frac{(\Omega^2 - n^2)\sin n\theta_p \sin\theta_p + n \sin n\theta_p \cos\theta_p}{\epsilon_n D_n(\Omega)} \\ &+ \sum_{n=0}^{\infty} \sin n\theta_p \sin\theta_p \frac{n \sin n\theta_p \sin\theta_p + (\Omega^2 - \beta^2 n^4 - 1)\sin n\theta_p \cos\theta_p}{\epsilon_n D_n(\Omega)}, \end{aligned} \tag{5.53}$$

$$\begin{aligned} N_{21} = &-\sum_{n=0}^{\infty} \cos n\theta_p \sin\theta_p \frac{(\Omega^2 - n^2)\cos n\theta_p \cos\theta_p - n \cos n\theta_p \sin\theta_p}{\epsilon_n D_n(\Omega)} \\ &- \sum_{n=0}^{\infty} \sin n\theta_p \cos\theta_p \frac{n \cos n\theta_p \cos\theta_p - (\Omega^2 - \beta^2 n^4 - 1)\cos n\theta_p \sin\theta_p}{\epsilon_n D_n(\Omega)}, \end{aligned} \tag{5.54}$$

and

$$\begin{aligned} N_{22} = &-\sum_{j=1}^{\infty} \frac{2\sin\theta_p}{M(\Omega^2 - \Omega^2_{(L)j})} \\ &- \sum_{n=0}^{\infty} \cos n\theta_p \sin\theta_p \frac{(\Omega^2 - n^2)\sin n\theta_p \sin\theta_p + n \sin n\theta_p \cos\theta_p}{\epsilon_n D_n(\Omega)} \\ &- \sum_{n=0}^{\infty} \sin n\theta_p \cos\theta_p \frac{n \sin n\theta_p \sin\theta_p + (\Omega^2 - \beta^2 n^4 - 1)\sin n\theta_p \cos\theta_p}{\epsilon_n D_n(\Omega)}. \end{aligned} \tag{5.55}$$

And the forcing terms are given by:

$$Q_1 = \sum_{n=0}^{\infty} \left[\cos n\theta_p \cos\theta_p \frac{(\Omega^2 - n^2)}{D_n(\Omega)} - \sin n\theta_p \sin\theta_p \frac{n}{D_n(\Omega)} \right] \tilde{p}_n \tag{5.56}$$

and

$$Q_2 = \sum_{n=0}^{\infty} \left[\cos n\theta_p \sin\theta_p \frac{(\Omega^2 - n^2)}{D_n(\Omega)} + \sin n\theta_p \cos\theta_p \frac{n}{D_n(\Omega)} \right] \tilde{p}_n . \tag{5.57}$$

This system of equations may be solved for the Lagrange multipliers in terms of the surface pressure to give:

$$\begin{aligned} \tilde{\lambda}_1 = \sum_{n=0}^{\infty} \Bigg(&\frac{\left[(\Omega^2 - n^2)\cos n\theta_p \cos\theta_p - n \sin n\theta_p \sin\theta_p\right] N_{22}}{D_n(\Omega)(N_{11}N_{22} - N_{12}N_{21})} \\ &- \frac{\left[(\Omega^2 - n^2)\cos n\theta_p \sin\theta_p + n \sin n\theta_p \cos\theta_p\right] N_{12}}{D_n(\Omega)(N_{11}N_{22} - N_{12}N_{21})} \Bigg) \tilde{p}_n \end{aligned} \tag{5.58}$$

and

$$\begin{aligned} \tilde{\lambda}_2 = \sum_{n=0}^{\infty} \Bigg(&\frac{\left[(\Omega^2 - n^2)\cos n\theta_p \sin\theta_p + n \sin n\theta_p \cos\theta_p\right] N_{11}}{D_n(\Omega)(N_{11}N_{22} - N_{12}N_{21})} \\ &- \frac{\left[(\Omega^2 - n^2)\cos n\theta_p \cos\theta_p - n \sin n\theta_p \sin\theta_p\right] N_{21}}{D_n(\Omega)(N_{11}N_{22} - N_{12}N_{21})} \Bigg) \tilde{p}_n . \end{aligned} \tag{5.59}$$

Equations (5.58) and (5.59) are now inserted back into Equation (5.46) to give an expression for $\tilde{U}_n$ in terms of the surface pressure. Comparing this result to Equations (4.8) and (4.9) immediately leads to the identification of the unknowns in Equation (4.9) as:

$$E_n = \frac{R_s^2}{h_s \rho_s c_s^2} \frac{(\Omega^2 - n^2)}{D_n(\Omega)}, \tag{5.60}$$

$$F_n = \frac{(\Omega^2 - n^2)\cos n\theta_p \cos\theta_p - n\cos n\theta_p \sin\theta_p}{\epsilon_n D_n(\Omega)}, \tag{5.61}$$

$$G_n = \frac{(\Omega^2 - n^2)\sin n\theta_p \sin\theta_p + n\sin n\theta_p \cos\theta_p}{\epsilon_n D_n(\Omega)}, \tag{5.62}$$

$$\begin{aligned} A_n = \frac{R_s^2}{h_s \rho_s c_s^2} \Bigg(& \frac{\left[(\Omega^2 - n^2)\cos n\theta_p \cos\theta_p - n\sin n\theta_p \sin\theta_p\right] N_{22}}{D_n(\Omega)(N_{11}N_{22} - N_{12}N_{21})} \\ & - \frac{\left[(\Omega^2 - n^2)\cos n\theta_p \sin\theta_p + n\sin n\theta_p \cos\theta_p\right] N_{12}}{D_n(\Omega)(N_{11}N_{22} - N_{12}N_{21})} \Bigg), \end{aligned} \tag{5.63}$$

and

$$\begin{aligned} B_n = \frac{R_s^2}{h_s \rho_s c_s^2} \Bigg(& \frac{\left[(\Omega^2 - n^2)\cos n\theta_p \sin\theta_p + n\sin n\theta_p \cos\theta_p\right] N_{11}}{D_n(\Omega)(N_{11}N_{22} - N_{12}N_{21})} \\ & - \frac{\left[(\Omega^2 - n^2)\cos n\theta_p \cos\theta_p - n\sin n\theta_p \sin\theta_p\right] N_{21}}{D_n(\Omega)(N_{11}N_{22} - N_{12}N_{21})} \Bigg). \end{aligned} \tag{5.64}$$

Based on this identification the elastically scattered pressure field is given (albeit in a rather complicated fashion) entirely in terms of known quantities through Equations (4.13) and (4.4).

§6. Closure

The boundary value problem for the scattering of a plane wave off an elastic shell with an internal plate in the steady state has been solved in terms of known quantities. The resulting expressions show the coupling of all the harmonics of the loading to each harmonic of the response. The solution also shows as is to be expected that excitations at the in vacuo natural frequencies of the shell or at the in vacuo natural frequencies of the longitudinal modes of the plate lead to strong far field radiation from the structure. Interestingly, though not surprisingly, excitations at the in vacuo natural frequencies of the transverse modes of the plate do not necessarily produce strong far field radiation. For the purpose of comparison and validation of other solution techniques, evaluation of the infinite sums involved in the solution can be accomplished via brute force by taking large numbers of terms or through use of the frequency window method of IGUSA, ACHENBACH, AND MIN [1991a,b].

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§8. References

Abramowitz, M. and Stegun, I.A., Eds. [1972], *Handbook of Mathematical Functions.* National Bureau of Standards, Washington, D.C.

Bjarnason, J., Achenbach, J.D., and Igusa, T. [1992], "Acoustic radiation from a cylindrical shell with an internal plate," *Wave Motion* **15,** 23-41.

Donnell, L.H. [1933], "Stability of Thin Walled Tubes Under Torsion," *N.A.C.A. Report No. 479.*

Igusa, T., Achenbach, J.D., and Min, K.W. [1991a], "Resonance Characteristics of Connected Subsystems: Theory and Simple Configurations," *J. Sound Vib.* **146,** 407-421.

Igusa, T., Achenbach, J.D., and Min, K.W. [1991b], "Resonance Characteristics of Connected Subsystems: General Configurations," *J. Sound Vib.* **146,** 423-437.

Junger, M.C. and Feit, D. [1972], *Sound, Structures, and Their Interaction.* The MIT Press, Cambridge, MA.

Kraus, H. [1967], *Thin Elastic Shells.* John Wiley, New York.

Szilard, R. [1974], *Theory and Analysis of Plates: Classical and Numerical Methods.* Prentice–Hall Inc., Englewood Cliffs, NJ.

Timoshenko, S. and Woinowsky-Krieger, S. [1959], *Theory of Plates and Shells.* McGraw–Hill, New York.

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