# *Viscoelastic Constitutive Relations for the Steady Spinning of a Cylinder*

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## § Abstract

In this paper we present a consistent viscoelastic model for examining the steady state spinning of a cylinder in contact with a frictional surface. The formulation of the problem is modeled after LE TALLEC & RAHIER [*Int. J. Numer. Meth. Engng.*, **37** 1159-1186 (1994)] and ODEN & LIN [*Comp. Meth. Appl. Mech. Engng.* **57** 297-367 (1986)]. A non-physical contribution to the viscoelastic effects of the stress is shown for the commonally advocated model. A methodology that immediately leads to an appropriate constitution given an objective viscoelastic model is presented. Numerical examples are presented showing significant differences between the viscoelastic stresses with respect to past works.

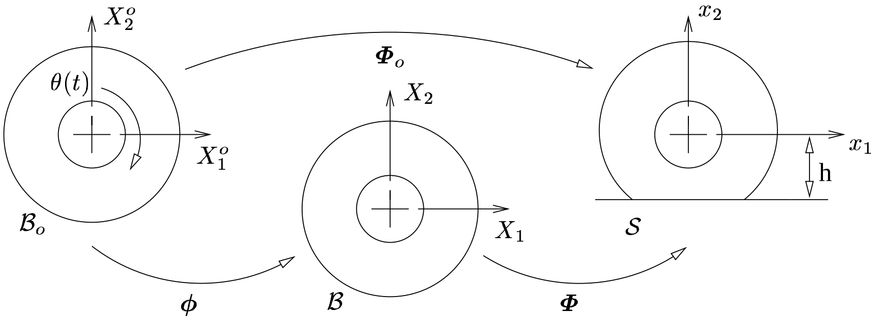


**FIGURE 1.1.** Moving reference frame

## §1. Introduction

The problem of interest consists of a deformable circular cylinder attached to a rigid circular core spinning with motion $\theta(t)$, see Figure **1.1**. The outer cylinder is then brought into contact with a flat frictional surface a distance $h$ from the center of spin. The material in the cylinder is assumed to be viscoelastic. An approach for arriving at a steady state solution is to recast this problem onto a moving reference frame. A technique of this nature in a finite element setting was proposed by Lynch [1] for the rolling of viscoelastic plates. For the spinning cylinder, formulations of varying degrees of sophistication have been proposed by many authors; see e.g. Oden & Lin [2], Le Tallec & Rahier [3], Padovan & Paramodilok [4], Padovan [5], Bass [6], Kennedy and Padovan [7], and other references therein. A crucial issue that seems to have been overlooked in the spinning cylinder problem is that the circular geometry affects the constitutive relations when expressed in the moving reference frame. This issue is not relevant to Lynch's [1] original work on flat plates by is important for all the subsequent work on the circular geometry. The recent formalism of Le Tallec & Rahier [3] makes this issue nearly transparent and provides an avenue for the easy specification of the correct constitutive relations on the moving reference frame. In fact in [3] a correct set of constitutive relations are utilized; though, these authors appear not to have recognized the significance of this outcome of their formulation. Here, we examine the the implications of this formalism on common convolution integral type models and present a quasi-Newton method for the solution of the weak form equations.

The paper is divided into four sections. Section 2 describes the kinematics of steady state spinning, states the equilibrium equations, and derives the weak form equations; Section 3 introduces the viscoelastic constitutive model and presents a method for solving the complete problem; in Section 4 a set of examples provide comparisons to the work of ODEN & LIN [2].

## §2. Kinematics and Equilibrium

Consider a point on a deformable cylinder spinning about a fixed axis. This point will naturally appear differently to an observer on the fixed axis as compared to an observer spinning with the cylinder. Specifically, the spinning observer will only see the strain of the point and not its rigid rotation. In solving the steady state spinning of a cylinder, we formulate the problem on a moving reference frame which is analogous to the rotating observer. For the spinning cylinder, the moving reference frame is a time dependent frame whose rotation about the axis of the cylinder is given by $\theta(t)$. For the steady state case, we can derive equilibrium equations with respect to the moving frame which are time independent. The final equations contain additional inertial terms associated with the moving frame. In choosing the presentation of the moving reference frame and the subsequent derivation of the weak form expressions, the notational simplifications of LE TALLEC & RAHIER [3] are used as they most clearly illustrate the needed constitutive relations.

A notational convention is first introduced for the moving frame. Next, we consider an application of this approach to the steady state spinning of a cylinder. The equilibrium equations and boundary conditions are stated on the fixed reference configuration and a set of weak form equations are derived. A recasting of the weak form equations onto the moving reference frame yields a complete statement of the balance equations for the steady state problem.

*2.1. Moving Reference Frame.*

Let the open set $\mathcal{B}_o \subset \mathbb{R}^3$ be the fixed reference placement of a continuum body containing the material points $\boldsymbol{X}_o \in \mathcal{B}_o$. Points in the fixed reference placement are mapped to the deformed configuration $\mathcal{S} \subset \mathbb{R}^3$ by the motion $\boldsymbol{x} = \boldsymbol{\Phi}_o(\boldsymbol{X}_o)$ where $\mathcal{S} = \boldsymbol{\Phi}_o(\mathcal{B}_o)$ and points in the deformed configuration are denoted by $\boldsymbol{x} \in \mathcal{S}$. Points in the moving reference configuration $\boldsymbol{X} \in \mathcal{B}$ are related to the fixed reference placement by the motion $\boldsymbol{X} = \boldsymbol{\phi}(\boldsymbol{X}_o)$ where $\mathcal{B} = \boldsymbol{\phi}(\mathcal{B}_o)$. The unknown motion of a particle with respect to the moving configuration is $\boldsymbol{x} = \boldsymbol{\Phi}(\boldsymbol{X}) = \boldsymbol{\Phi}_o \circ \boldsymbol{\phi}^{-1}(\boldsymbol{X})$. We define the deformation gradient with respect to the moving reference placement as $\boldsymbol{F} = \mathrm{GRAD}(\boldsymbol{\Phi})$ where $\mathrm{GRAD}(\cdot)$ denotes the gradient with respect to $\boldsymbol{X}$. The associated "right Cauchy-Green" deformation tensor is given by $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$.

*2.2. Steady State Spinning of a Cylinder.*

Define the moving reference configuration of a spinning cylinder by the motion

$$\boldsymbol{\phi}(\boldsymbol{X}_o) = \boldsymbol{R}\boldsymbol{X}_o, \tag{2.1}$$

where $\boldsymbol{R}$ is a time dependent rotation $\theta$ about the axis of the cylinder. For this motion, the deformation measures with respect to the fixed reference placement become

$$\boldsymbol{F}_o = \boldsymbol{F}\boldsymbol{R} \quad \text{and} \quad \boldsymbol{C}_o = \boldsymbol{R}^T\boldsymbol{C}\boldsymbol{R}, \tag{2.2}$$

where all quantities with a subscript o are with respect to the fixed reference placement.

The velocity and acceleration of a particle $\boldsymbol{x}$ in the current configuration for the steady spinning case can be computed as

$$\dot{\boldsymbol{x}} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\boldsymbol{\Omega}\boldsymbol{X} \tag{2.3}$$

and

$$\ddot{\boldsymbol{x}} = (\frac{\partial^2 \boldsymbol{x}}{\partial \boldsymbol{X}^2}\boldsymbol{\Omega}\boldsymbol{X} + \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\boldsymbol{\Omega})\boldsymbol{\Omega}\boldsymbol{X}, \tag{2.4}$$

where $\boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^T$ is the time independent skew-symmetric spin tensor associated with the mapping from the fixed reference configuration to the moving reference frame.

*2.3. Equilibrium.* The boundary value problem for the unknown motion $\boldsymbol{\Phi}_o$ with respect to the fixed reference frame is defined by the following equilibrium equations and boundary conditions; for all $\boldsymbol{X}_o \in \mathcal{B}_o$,

$$\text{DIV}_o[\boldsymbol{F}_o\boldsymbol{S}_o] + \rho_o\widehat{\boldsymbol{b}} = \rho_o\ddot{\boldsymbol{x}} \qquad \text{and} \qquad \boldsymbol{S}_o = \boldsymbol{S}_o^T \tag{2.5}$$

for all $\boldsymbol{X}_o \in \partial\mathcal{B}_o^\sigma$

$$\boldsymbol{F}_o\boldsymbol{S}_o\boldsymbol{N}_o = \bar{\boldsymbol{p}} \tag{2.6}$$

and for all $\boldsymbol{X}_o \in \partial\mathcal{B}_o^{\Phi_o}$

$$\boldsymbol{\Phi}_o = \bar{\boldsymbol{\Phi}}_o, \tag{2.7}$$

where $\boldsymbol{S}_o$ is the 2nd Piola-Kirchhoff stress tensor, $\mathrm{DIV}_o[\cdot]$ is the divergence operator with respect to $\boldsymbol{X}_o$, $\widehat{\boldsymbol{b}}$ is a given body force per unit mass, $\bar{\boldsymbol{p}}$ is a given traction function per unit reference area, $\boldsymbol{N}_o$ is the reference surface normal, $\bar{\boldsymbol{\Phi}}_o$ is a given surface motion, $\partial\mathcal{B}_o^\sigma \cap \partial\mathcal{B}_o^{\Phi_o} = \emptyset$, and $\overline{\partial\mathcal{B}_o^\sigma \cup \partial\mathcal{B}_o^{\Phi_o}} = \partial\mathcal{B}_o$ the boundary of $\mathcal{B}_o$.

*2.4. Weak Form and Linearization.* The weak form equations for the equilibrium problem are formed by multiplying (2.5) by arbitrary admissible weighting functions, integrating over the fixed reference domain, and performing integration by parts on the result. The resulting weak form expression is,

$$G_o(\boldsymbol{\Phi}_o;\boldsymbol{\eta}) = \int_{\mathcal{B}_o}(\boldsymbol{F}_o\boldsymbol{S}_o) : \frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{X}_o} - \int_{\mathcal{B}_o}\rho_o\widehat{\boldsymbol{b}}\cdot\boldsymbol{\eta} - \int_{\partial\mathcal{B}_o^\sigma}\bar{\boldsymbol{p}}\cdot\boldsymbol{\eta} + \int_{\mathcal{B}_o}\rho_o\ddot{\boldsymbol{x}}\cdot\boldsymbol{\eta} = 0, \tag{2.8}$$

where the admissible weighting functions $\boldsymbol{\eta} : \mathcal{B}_o \longrightarrow \mathbb{R}^3$ and $\boldsymbol{\eta} = 0$ on $\partial\mathcal{B}_o^{\Phi_o}$. Recasting (2.8) onto the moving reference frame and substituting in (2.2a) and the steady state expression for the acceleration (2.4) yields,

$$\begin{aligned} G(\boldsymbol{\Phi};\boldsymbol{\eta}) = & \int_{\mathcal{B}}(\boldsymbol{F}\boldsymbol{S}) : \frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{X}} - \int_{\mathcal{B}}\rho(\widehat{\boldsymbol{b}} - \boldsymbol{\Omega}^2\boldsymbol{X})\cdot\boldsymbol{\eta} \\ & - \int_{\partial\mathcal{B}_\sigma}\bar{\boldsymbol{p}}\cdot\boldsymbol{\eta} - \int_{\mathcal{B}}\rho(\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{X}}\boldsymbol{\Omega}\boldsymbol{X})\cdot(\frac{\partial\boldsymbol{\eta}}{\partial\boldsymbol{X}}\boldsymbol{\Omega}\boldsymbol{X}) = 0, \end{aligned} \tag{2.9}$$

where $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ is the displacement field with respect to the moving frame and $\boldsymbol{S}$ is *defined* to be

$$\boldsymbol{S} = \boldsymbol{R}\boldsymbol{S}_o\boldsymbol{R}^T. \tag{2.10}$$

In arriving at this result we have assumed that $\dot{\boldsymbol{X}}\cdot\boldsymbol{N} = (\boldsymbol{\Omega}\boldsymbol{X})\cdot\boldsymbol{N} = 0$ on the outer boundary. Thus, (2.9) applies only when the boundary is circular.

**Remark 2.1.**

For completeness, we note the contribution to the linearization of (2.9) by the terms associated with the moving frame. Linearization of (2.9) in the arbitrary direction $\boldsymbol{\nu} : \mathcal{B} \longrightarrow \mathbb{R}^3$; where $\boldsymbol{\nu} = 0$ on $\partial\mathcal{B}_\phi$ yields the tangent

$$\boldsymbol{K} = D_1 G(\boldsymbol{\Phi}; \boldsymbol{\eta})[\boldsymbol{\nu}] = \tilde{\boldsymbol{K}} - \int_{\mathcal{B}} \rho(\frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{X}} \boldsymbol{\Omega} \boldsymbol{X}) \cdot (\frac{\partial \boldsymbol{\nu}}{\partial \boldsymbol{X}} \boldsymbol{\Omega} \boldsymbol{X}), \tag{2.11}$$

where $\tilde{\boldsymbol{K}}$ is the "standard tangent." □

**Remark 2.2.**

The tremendous utility of (2.9) stems from the fact that one can create a finite element model of the cylinder in the moving frame and compute the deformation relative to it. Thus the steady state problem becomes, essentially, a static analysis. □

## §3. Constitution

The final ingredient needed to effect a stress analysis using (2.9) is a constitutive relation. Based on Le Tallec & Rahier's [3] notational scheme which leads to (2.9), one sees that the proper expression for the stress $\boldsymbol{S}$ is a rotated version of a standard fixed frame constitutive relation for the second Piola-Kirchhoff stress $\boldsymbol{S}_o$.

*3.1. Isotropic Elastic Materials.* Remarkably for isotropic elastic solids the expression for $\boldsymbol{S}$ can be given in a form "independent" of $\boldsymbol{R}(t)$. Using the Representation Theorem (see e.g. GURTIN [8, §37]) we can write

$$\boldsymbol{S}_o = k_1 \boldsymbol{C}_o + k_2 \boldsymbol{C}_o^2 + k_3 \mathbf{1}, \tag{3.1}$$

where $k_1$, $k_2$, and $k_3$ are functions of the invariants of $\boldsymbol{C}_o$; that is to say $\mathrm{I}_{C_o} = \mathrm{tr}\,[\boldsymbol{C}_o]$, $\mathrm{II}_{C_o} = 1/2[(\mathrm{tr}\,(\boldsymbol{C}_o))^2 - \mathrm{tr}\,(\boldsymbol{C}_o^2)]$, and $\mathrm{III}_{C_o} = J_o = \det[\boldsymbol{C}_o]$. Noting now that $\boldsymbol{S} = \boldsymbol{R}\boldsymbol{S}_o\boldsymbol{R}^T$, we have

$$\boldsymbol{S} = k_1 \boldsymbol{C} + k_2 \boldsymbol{C}^2 + k_3 \mathbf{1}, \tag{3.2}$$

where $k_1$, $k_2$, and $k_3$ are taken as functions of the invariants $\mathrm{I}_C$, $\mathrm{II}_C$, and $\mathrm{III}_C$ since the invariants of $\boldsymbol{C}$ and $\boldsymbol{C}_o$ are identical. Thus $\boldsymbol{S}$ is given entirely in terms of the deformation from the rotating frame. Knowledge of the true reference configuration is not needed. Note further that the final expression (3.2) merely involves removing the subscript o's from all terms in the "standard expression" (3.1).

*3.2. Isotropic Viscoelastic Constitution.* In the classical work of ODEN & LIN [2], a viscoelastic model for $\boldsymbol{S}$ was provided by taking the model of CHRISTENSEN [9] for a fixed reference frame and effectively "dropping all the subscript o's." A similar situation can also be observed in [4-7] (though not as clearly, since not all kinematic and stress quantities that appear were fully defined in these works). Christensen's original model can be written in the present notation as:

$$\boldsymbol{S}_o(t) = \widehat{\boldsymbol{S}}^e(\boldsymbol{E}_o(t)) + \nu \int_{-\infty}^{t} \exp\left(-(t-s)/\tau\right) \frac{\partial \boldsymbol{E}_o}{\partial s} ds, \tag{3.3}$$

where t is the current time, $\boldsymbol{E}_o = 1/2(\boldsymbol{C}_o - \mathbf{1})$, $\widehat{\boldsymbol{S}}^e(\cdot)$ is the Mooney-Rivlin response function, and $\nu$ and $\tau$ are material parameters. In [2] it is proposed that

$$\boldsymbol{S}(t) = \widehat{\boldsymbol{S}}^e(\boldsymbol{E}(t)) + \nu \int_{-\infty}^{t} \exp\left(-(t-s)/\tau\right) \frac{\partial \boldsymbol{E}}{\partial s} ds \tag{3.4}$$

in the steady state spinning problem. Is is noted that such a relation is inconsistent with the development leading to the weak form (2.9). Using the proper relation for $\boldsymbol{S}$, (2.10), and (3.3) shows

$$\boldsymbol{S}(t) = \widehat{\boldsymbol{S}}^e(\boldsymbol{E}(t)) + \nu \int_{-\infty}^{t} \exp\left(-(t-s)/\tau\right) \frac{d}{ds} \left\{\boldsymbol{R}(t)\boldsymbol{R}^T(s)\boldsymbol{E}(s)\boldsymbol{R}(s)\boldsymbol{R}^T(t)\right\} ds, \qquad (3.5)$$

where $\boldsymbol{E} = 1/2(\boldsymbol{C} - \mathbf{1})$. The error involved in using (3.4) versus (3.5), involves a non-physical contribution of viscoelastic effects to the stress. In particular the orientation of the "over-stress" tensor is not properly accounted for in (3.4).

**Remark 3.1.**

When using (3.5) it is noted that even though $\boldsymbol{R}(\cdot)$ explicitly appears the entire motion need not be known since $\boldsymbol{R}(\cdot)$ always appears at $\boldsymbol{R}(t)\boldsymbol{R}^T(s)$; i.e. only the relative rotation between two points in time is needed. Due to the steady nature of the problem this is explicitly give by $\exp[\boldsymbol{\Omega}(t-s)]$. □

If we consider the more general N-relaxation mechanism model of Govindjee & Simo [10] we obtain similar results. In particular the fixed reference frame model is given by

$$\boldsymbol{S}_o(t) = 2\nabla W^e(\boldsymbol{C}_o(t)) + \mathrm{DEV}_o(t)\left[\sum_{k=1}^{N} \boldsymbol{Q}_o^k(t)\right], \qquad (3.6)$$

where $W^e$ is the "long-time" hyper-elastic stored energy function for the material, $\mathrm{DEV}_o(\cdot) = J_o^{-2/3}\{(\cdot) - 1/3[(\cdot) : \boldsymbol{C}_o]\boldsymbol{C}_o^{-1}\}$, and $\boldsymbol{Q}_o^k$ are a set of viscoelastic second Piola-Kirchhoff over-stresses. More specifically

$$\boldsymbol{Q}_o^k(t) = \int_{-\infty}^{t} \exp\left(-(t-s)/\tau_k\right) \frac{d}{ds} \left\{\mathrm{DEV}_o(s)\left[2\nabla W^k(\boldsymbol{C}_o(s))\right]\right\} ds, \qquad (3.7)$$

where $\tau_k$ are the relaxation times and the $W^k(\cdot)$ are stored energy functions associated with each relaxation mechanism. In this setting, if $W^e(\cdot)$ and $W^k(\cdot)$ are isotropic, then one has that

$$\boldsymbol{S}(t) = 2\nabla W^e(\boldsymbol{C}(t)) + \mathrm{DEV}(t)\left[\sum_{k=1}^{N} \boldsymbol{Q}^k(t)\right], \tag{3.8}$$

where $\mathrm{DEV}(\cdot) = J^{-2/3}\{(\cdot) - 1/3[(\cdot):\boldsymbol{C}]\boldsymbol{C}^{-1}\}$ and

$$\boldsymbol{Q}^k(t) = \int_{-\infty}^{t} \exp\left(-(t-s)/\tau_k\right) \frac{d}{ds}\left\{\boldsymbol{R}(t)\boldsymbol{R}^T(s)\mathrm{DEV}\left[2\nabla W^k(\boldsymbol{C}(s))\right]\boldsymbol{R}(s)\boldsymbol{R}^T(t)\right\} ds. \tag{3.9}$$

Thus in this more general setting one also has the result that the appropriate stress for the weak form can be computed without a knowledge of the entire motion of the cylinder.

*3.3. Viscoelastic Steady State Space-Time Mapping.* The viscoelastic model requires a history of the particle strains over time. For the steady state problem, the time domain can be held fixed and computations based on a particles history are mapped onto an angular position as proposed first by LYNCH [1]. In other words the history data at a past time $s$ for a material particle currently at say an angle $\beta$ can be found in the steady case by looking at the state of the material located (at the current time $t$) at the angle $\beta - ||\boldsymbol{\Omega}||(t-s)$. Thus, the history of a particle is shown to be directly related to the annulus of particles within a fixed radius in the moving reference frame. Applying this result to the viscoelastic contribution (3.9) gives:

$$\boldsymbol{Q}^k(\beta) = \mathrm{DEV}(\beta)\left[2\nabla W^k(\boldsymbol{C}(\beta))\right] - \frac{1}{||\boldsymbol{\Omega}||\tau_k}\left\{\int_{-\infty}^{\beta} \exp\left(-\frac{\beta-\alpha}{||\boldsymbol{\Omega}||\tau_k}\right)\boldsymbol{R}(\beta)\boldsymbol{R}^T(\alpha)\mathrm{DEV}(\alpha)\left[2\nabla W^k(\boldsymbol{C}(\alpha))\right]\boldsymbol{R}(\alpha)\boldsymbol{R}^T(\beta)d\alpha\right\}. \tag{3.10}$$

### *3.4. Viscoelastic BVP solution.*

Relation (2.9) can in principle be solved by a straight-forward Newton method. However since (3.10) yields a non-local constitution the linearization of the viscous stresses is non-standard and does not easily fit typical FEM code architectures. Thus we propose a quasi-Newton methodology where the viscoelastic stresses $\mathrm{DEV}(t)\left[\sum_{k=1}^{N}\boldsymbol{Q}^k(t)\right]$, are held fixed during a series of Newton iterations on (2.9). They are then updated based on the final iterate through an integration of (3.10) and a new Newton sequence is initiated. This proceeds until convergence.

A one point Euler integration rule is used to integrate (3.10),

$$\boldsymbol{Q}^k(\beta) = \mathrm{DEV}(\beta)[2\nabla W^k(\beta)] - \frac{\Delta\beta}{||\boldsymbol{\Omega}||\tau_k}\left[\sum_{i=1}^{\bar{N}}\exp\left(-\frac{\beta-\beta_i}{||\boldsymbol{\Omega}||\tau_k}\right)\boldsymbol{R}(\beta)\boldsymbol{R}^T(\beta_i)\mathrm{DEV}(\beta_i)\left[2\nabla W^k(\boldsymbol{C}(\beta_i))\right]\boldsymbol{R}(\beta_i)\boldsymbol{R}^T(\beta)\right], \tag{3.11}$$

where $\beta_i = \beta - i\Delta\beta$, $\Delta\beta$ is the angular resolution of the mesh, and $\bar{N}$ is the number of points used in the Euler approximation. Review of this equation shows that the viscoelastic overstress for a point is a weighted sum of all previous points within an annulus containing the point of interest. Also, the weighting of the sum has an exponential decay which may require multiple revolutions around the annulus before it decays sufficiently. Thus, $\bar{N}$ should be chosen to include all terms which yield a significant contribution to the viscoelastic stress. We note that each term in the series sum can be mapped directly to a circumferential element in the finite element model.

## §4. Illustrations

In this section we provide comparisons to the work of ODEN & LIN [2] with particular emphasis on comparing (3.4) and (3.5). These results will illustrate the necessity for a consistently formulated viscoelastic model.

We assume 2D plane strain and use a Mooney-Rivlin constitutive relationship $\tilde{W}^{\infty}(\tilde{\boldsymbol{C}}) = C_1(I_{\tilde{C}} - 3) + C_2(II_{\tilde{C}} - 3)$ with material constants $C_1 = 80$ (psi) and $C_2 = 20$ (psi). The total strain energy function is $W^e(\boldsymbol{C}_o) = \tilde{W}^{\infty}(\tilde{\boldsymbol{C}}_o) + U(J_o)$, where $\tilde{\boldsymbol{C}}_o = J_o^{-2/3}\boldsymbol{C}_o$, $U(J_o) = \frac{1}{2}\kappa(J_o - 1)^2$, and $\kappa$ is the bulk modulus. A two field pressure-displacement formulation is used for the calculation of the problem with $\kappa = 1.0 \times 10^5$ (psi) to enforce an incompressibility constraint; see SUSSMAN & BATHE [11]. The density of the material is $\rho = 0.036$ (lb-sec$^2$/in$^4$). The constitutive parameter $\tau = 0.1$ (sec) and $\nu = 100$ (psi). The integration of the convolution integrals in (3.4) and (3.5) is performed in a manner analogous to (3.11). To insure enough terms were included in the summation, we set $\bar{N}$ such that 2 revolutions around the cylinder are considered.

We consider a problem definition that is identical to ODEN & LIN [2], with inner radius $R_i = 1$ (in), outer radius $R_o = 2$ (in). A rigid frictional surface is raised such that $h = .8$ (in) in Figure **1.1**. The moving reference frame motion is defined by $\theta(t) = ||\boldsymbol{\Omega}||t$ where the rate of spin $||\boldsymbol{\Omega}|| = 10$ (rad/sec). In this work a 4-node bilinear element with constant pressure is used. This is in contrast to a 9-node biquadratic element with linear pressure used in ODEN & LIN [2]. To more accurately represent this higher order element, a finer discretization of the mesh was employed with twice the circumferential and radial elements. The finite element mesh constructed contains 96 circumferential elements and 6 radial elements. The Newton Raphson Method is used to solve (2.9) with tangent

(2.11). Upon convergence of the Newton Raphson Method, the viscoelastic update (3.11) is employed. This process is repeated until no more Newton Raphson iterations are needed to converge the solution following a viscoelastic update.

The problem was solved in 20 load increments with each increment requiring 7 viscoelastic updates. The first increment required 6 Newton iterations which reduced the residual by 12 orders of magnitude; this was typical of other load increments. The polar components of the stress $\boldsymbol{S}$ at the Gauss points closest to the outer surface are reported in Figure 4.1. In the figures the abscissa measures angle in degrees clockwise starting from $(X_1, X_2) = (2, 0)$. We note that the stresses using Eq. (3.4) fall close to those reported in [2] but are not exactly the same due to the different element formulation used here. To clearly illustrate the necessity for a consistent stress formulation, we include only the viscoelastic contribution to the stresses in Figure 4.2. We see that these results are very dissimilar. The similarities in Figure 4.1 can be attributed to the fact that the results for this particular boundary value problem are dominated by elastic stresses.

**Remark 4.1.**

For the contact and regularized friction we have utilized the formulation and properties given by ODEN & LIN [2]. This method requires calculation of the slip velocity to determine the friction forces. The slip velocity is,

$$w_t = v_1 + ||\boldsymbol{\Omega}||h. \tag{4.1}$$

where $v_1$ is the 1-component of the velocity of the material on the contact surface. In our implementation, we have used 3-node contact elements, where nodes 2 and 3 are surface nodes on the cylinder and node 1 is a fixed reference node on the flat contact surface. The velocity $v_1$ is determined using the following mid-point approximation (superscripts denote node numbers):

$$v_1 = ||\boldsymbol{\Omega}||\frac{\partial \boldsymbol{u}}{\partial \theta} \cdot \boldsymbol{E}_1 \approx ||\boldsymbol{\Omega}||(u_1^2 - u_1^3)\left[\frac{X_2}{(X_1^2 - X_1^3)} - \frac{X_1}{(X_2^2 - X_2^3)}\right] + X_2||\boldsymbol{\Omega}||,$$

where $X_1 = 1/2(X_1^2 + X_1^3)$ and $X_2 = 1/2(X_2^2 + X_2^3)$. $\square$

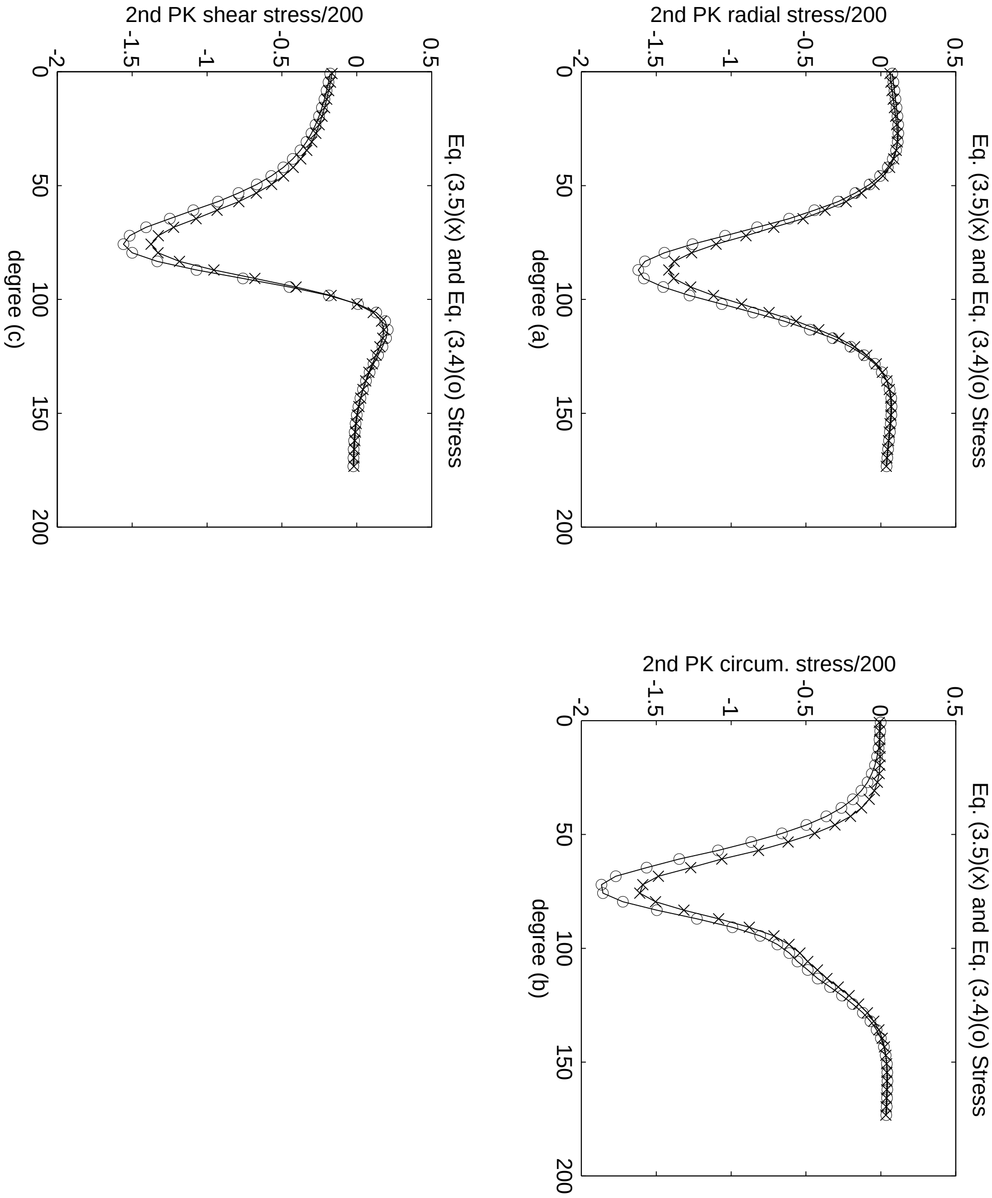


**FIGURE 4.1.** Comparisons of the components of the total stress near the outer surface: (a) radial, (b) circumferential, and (c) shear for Eqs. (3.4) and (3.5).

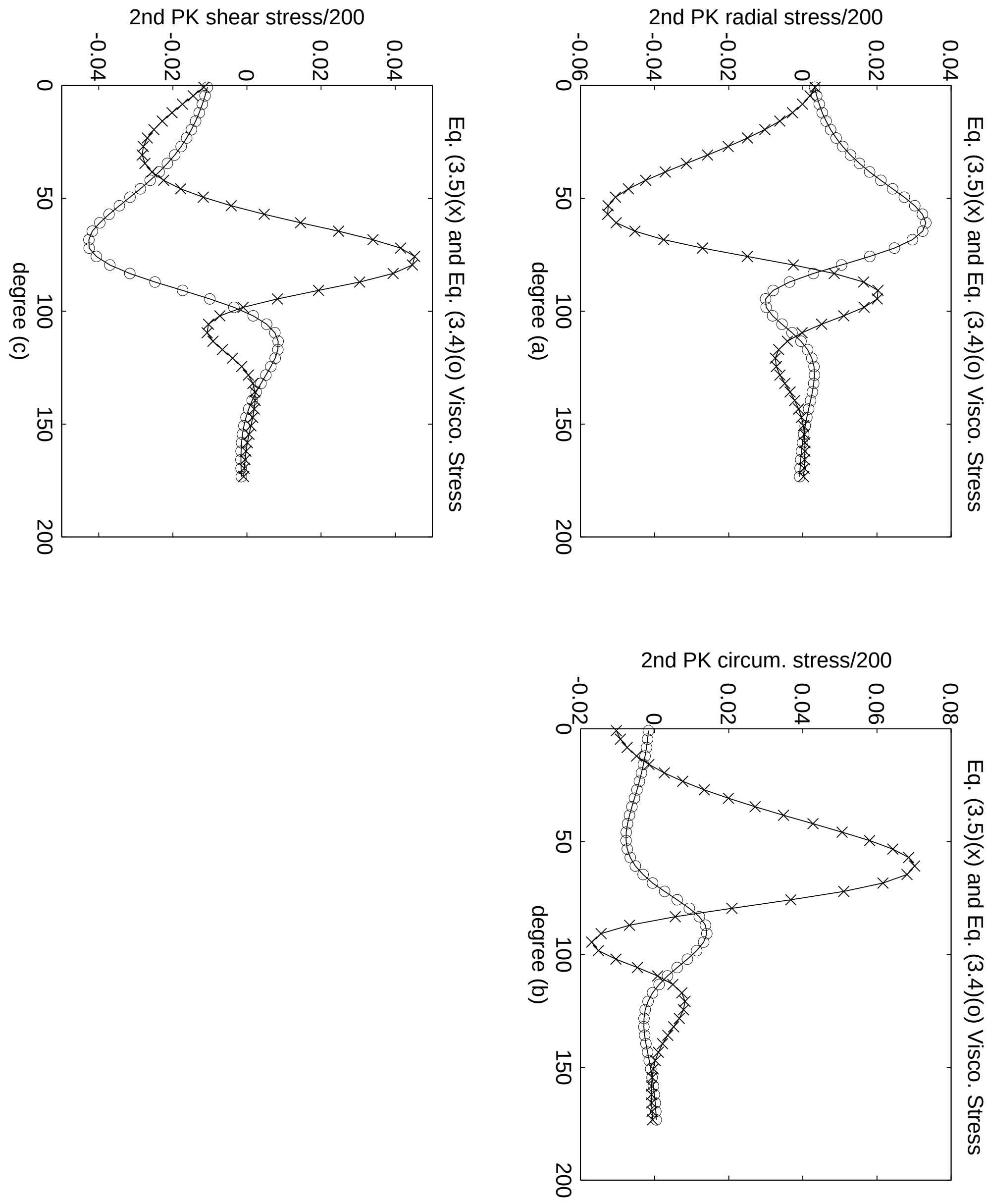


**FIGURE 4.2.** Comparisons of the components of the viscoelastic "over-stress" near the outer surface: (a) radial, (b) circumferential, and (c) shear for Eqs. (3.4) and (3.5).

## §5. Closure

In this paper we have presented a formulation of steady state rolling of a viscoelastic cylinder using the formalism of LE TALLEC & RAHIER [3] in conjunction with two convolution models for finite deformation (linear) viscoelasticity. It has been shown that the proper expressions for the stresses on the moving reference frame involve the addition of rotational terms that account for the convection of the material as the cylinder rotates. The importance of the correction involved in using these rotational terms has been demonstrated by looking at an example problem originally given in [2].

It is remarked in closing that the importance of the rotational terms in the constitutive relations depends strongly on the geometry, speed of rotation, and material properties. For instance if the radius of the cylinder is large and the relaxation times small in comparison to the rotational speed, then the contribution to the viscoelastic stresses will only come from points near the contact surface. For small contact areas, one could then ignore the rotational terms as they will be close to the identity.

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