Vainberg's Theorem and Corollary

Consider the weak form problem:

Find $u \in \mathcal{S}$ such that $B(w, u) - l(w) = 0$ for all $w \in \mathcal{V}$.

In class we discussed that if the functional $B(\cdot, \cdot)$ was bi-linear symmetric and $l(\cdot)$ was linear, then it was possible to phrase the weak form problem as a minimization problem; viz. minimize

$$I(u) = \frac{1}{2} B(u, u) - l(u) \tag{1}$$

over the space $\mathcal{S}$ – modulo a few mathematical technicalities. While this is true, the statement is more restrictive than need be. The necessary and sufficient condition for the existence of a minimization form of the problem (leaving aside some technical details) is that the *variation* of $B(\cdot, \cdot)$ with respect to u has a particular symmetry property. Consider $B(w_1, u)$ which is by construction linear in its first argument. Now define its variation in its second argument as

$$\delta B(w_1, w_2, u) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} B(w_1, u + \epsilon w_2)\,, \tag{2}$$

where w_2, like w_1, is an element of $\mathcal{V}$ the space of test functions. If

$$\delta B(w_1, w_2, u) = \delta B(w_2, w_1, u)\,, \tag{3}$$

then a minimization form for the problem exists and in fact the needed functional is given by

$$I(u) = I(u_o) + \int_0^1 B(u - u_o, u_o + t(u - u_o)) - l(u - u_o) dt\,, \tag{4}$$

where u_o is some admissible solution; i.e. $u_o \in \mathcal{S}$. These two results are the contents of a celebrated theorem and corollary of M.M. Vainberg (§5.3).

References

1. M.M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, Translated: A. Feinstein, Holden-Day (San Francisco), 1964.