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Notes on 1D isoparametric elements

1 Isoparametric Concept

The isoparametric concept in one-dimension is a method of standardizing the necessary computations to build the stiffness matrix and the righthand side forcing vector:

$$k^e_{ij} = \int_{\Omega_e} \frac{dN^e_i}{dx} AE \frac{dN^e_j}{dx}\, dx \tag{1}$$

$$f^e_i = \int_{\Omega_e} N^e_i b\, dx\,. \tag{2}$$

Here $N^e_i(x)$ is the shape function. In the basic set up, one uses Lagrange interpolation polynomials for these functions and the unknown $u^h(x) = \sum_i u^h_i N^e_i(x)$ for $x \in \Omega_e$.

In the isoparametric setting we first define shape functions over the fixed parent domain $[-1, 1]$. These functions are denoted as $N_i(\xi)$ and are determined by way of the Lagrange interpolation formulae applied to the parent element. In the parent element the nodes are always equally spaced. To be able to evaluate (1) and (2) we need, however, $N^e_i(x)$. In the isoparametric setting these are defined by postulating a connection between the parent element and the physical element. The connection is known as the isoparametric map, $x^e(\xi)$; we place the superscript e on the x to remind us that the mapping is element-by-element. Graphically the map is simply a point mapping from $[-1, 1]$ to Ω_e; see Fig. 1 for the quadratic isoparametric mapping case. Mathematically, the isoparametric map for an

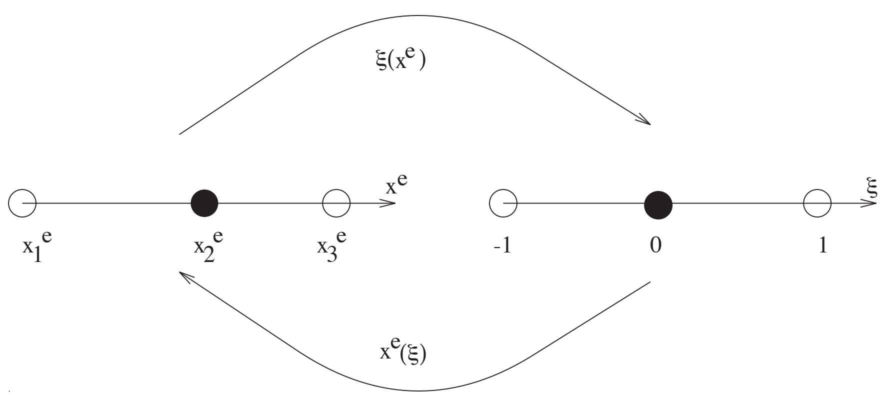

Figure 1: Isoparametric mapping, $x(\xi)$, for the quadratic element.

element e is defined by the relation

$$x^e(\xi) := \sum_i x_i^e N_i(\xi)\,. \tag{3}$$

The requirement on the mapping is that it be one-to-one and hence invertible. For us this can be represented by the requirement $j^e := dx^e/d\xi > 0$ for $\xi \in (-1,1)$. With (3) defined we can now define the shape functions over the physical domain as

$$N_i^e(x^e) = N_i(\xi(x^e)) = N_i(\xi) \circ \xi(x^e)\,. \tag{4}$$

Remarks

1. Note that (4) employs the inverse of the isoparametric map, hence the earlier statement that the map needs to be invertible.

2. If the isoparametic map turns out to be linear for a particular element, then shape functions (in (4)) over the physical element turn out to just be our original Lagrange polynomial shape functions. However is the mapping is not linear, then the shape functions in (4) are slightly different.

3. In the case where the shape functions over the physical element are no longer Lagrange polynomials, the primary requirement for our error estimates to still hold is that they be able to exactly represent arbitrary linear functions over a single element (when dealing with problems with at most first derivatives in the weak form). This is often call the completeness requirement (for problems with single derivatives in the weak form).

4. Observe that the unknown field over a single element is $u^h(x^e) = \sum_i u_i^h N_i^e(x^e)$. When parametrized over the parent element this expression becomes $u^h(\xi) = \sum_i u_i^h N_i(\xi)$, which has the exact same form as (3). This is the origin of the terminology: *iso*parametric. The parameterization of the geometry and unknown field is the same.

1.1 Consequence

If we use the relations outlined above and chase through the chain rule we come to the final expressions

$$k_{ij}^e = \int_{[-1,1]} \frac{dN_i}{d\xi}(\xi) AE \frac{dN_j}{d\xi}(\xi) \frac{1}{j^e}\, d\xi \tag{5}$$

$$f_i^e = \int_{[-1,1]} N_i(\xi) b\, j^e\, d\xi\,. \tag{6}$$

All the entries are standard/uniform for all elements. The element geometry is completely contained in the *Jacobian:* j^e.

1.2 Example: Linear Isoparametric Element

Consider a generic linear element with nodes at x_1^e and x_2^e. The isoparametric shape functions over the parent domain are given by

$$N_1(\xi) = \frac{1-\xi}{2} \qquad N_2(\xi) = \frac{1+\xi}{2} \tag{7}$$

and the shape function derivatives are given by

$$\frac{dN_1}{d\xi}(\xi) = -\frac{1}{2} \qquad \frac{dN_2}{d\xi}(\xi) = \frac{1}{2}\,. \tag{8}$$

The element Jacobian is given by

$$j^e = \frac{dx^e}{d\xi} = x_1^e \frac{dN_1}{d\xi}(\xi) + x_2^e \frac{dN_2}{d\xi}(\xi) = \frac{x_2^e - x_1^e}{2} = \frac{h^e}{2}\,, \tag{9}$$

where h^e is the physical length of the element. Assuming that b and AE are constants, this delivers the result that for all elements

$$k_{ij}^e = \frac{AE}{h^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{10}$$

$$f_i^e = \frac{bh^e}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\,. \tag{11}$$

1.3 Example: Quadratic Isoparametric Element

Consider a quadratic isoparametric element with nodes at x_1^e, $x_2^e = (x_1^e + x_3^e)/2$ and x_3^e. Note the uniform spacing of the physical nodes should give us a linear isoparametric map. (In the next example we will treat the case where the physical nodes are not uniformly spaced.) In this case the nodes in the parent domain are located at -1, 0, and 1. The resulting shape functions over the parent domain are

$$N_1(\xi) = \frac{1}{2}\xi(\xi-1) \qquad N_2(\xi) = (1-\xi)(1+\xi) \qquad N_3(\xi) = \frac{1}{2}\xi(\xi+1)\,. \tag{12}$$

The derivatives of the shape functions are

$$\frac{dN_1}{d\xi} = \xi - 1/2 \qquad \frac{dN_2}{d\xi} = -2\xi \qquad \frac{dN_3}{d\xi} = \xi + 1/2\,. \tag{13}$$

Using the given (uniform) nodal spacings, the isoparametric map is linear:

$$\begin{aligned} x^e(\xi) &= \sum_{i=1}^{3} x_i^e N_i(\xi) = x_1^e \frac{1}{2}\xi(\xi-1) + \frac{x_1^e + x_3^e}{2}(1-\xi^2) + x_3^e \frac{1}{2}\xi(\xi+1) \\ &= \left(x_1^e \frac{1}{2} - \frac{x_1^e + x_3^e}{2} + x_3^e \frac{1}{2}\right)\xi^2 + \frac{x_3^e - x_1^e}{2}\xi + \frac{x_1^e + x_3^e}{2} \\ &= \frac{h^e}{2}\xi + \frac{x_1^e + x_3^e}{2} \end{aligned} \tag{14}$$

Plugging in we find

$$k_{ij}^e = \frac{AE}{3h^e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \tag{15}$$

$$f_i^e = \frac{bh^e}{3}\begin{pmatrix} 1/2 \\ 2 \\ 1/2 \end{pmatrix} . \tag{16}$$

Remarks

1. In these two examples the isoparametric map is linear and thus the resulting (physical domain) shape functions are just the Lagrange shape functions.

1.4 Example: Quadratic Isoperimetric Shape Functions

To appreciate the last remark, consider the case of $x_1^e = 0.0$, $x_2^e = 0.6$, and $x_3^e = 1.0$. The spacing of the physical nodes is no longer uniform. In this case,

$$x^e(\xi) = \xi(1+\xi)/2 + 0.6(1-\xi^2) \tag{17}$$

and the inverse function is given by

$$\xi(x) = 0.5\left(5.0 - \sqrt{49.0 - 40.0x^e}\right) . \tag{18}$$

If one plots the standard Lagrange shape functions over this element and compares them to the isoparametric ones over the same physical element (i.e. plot (4)), one sees that they differ from each other. Both sets possess the Kronecker property but they are slightly different from each other.

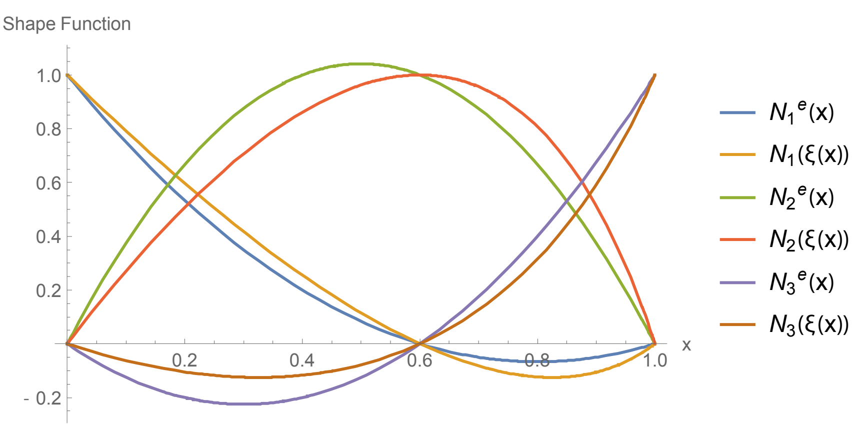

Figure 2: Isoparametric shape functions versus Lagrange shape functions when $x_2^2 \neq (x_1^e + x_3^e)/2$. Case shown is for $x_1^e = 0.0$, $x_2^e = 0.6$, and $x_3^e = 1.0$

The difference in the shape functions also results in different element force vector and stiffness matrix. For the Lagrange case applied to this element one has the results shown in (15) and (16) with $h^e = 1$. If however we use the isoparametric shape functions over this element one finds

$$k^e_{ij} = \frac{AE}{3} \begin{bmatrix} 5.8 & -7.1 & 1.3 \\ -7.1 & 18 & -11 \\ 1.3 & -11 & 9.4 \end{bmatrix} \tag{19}$$

$$f^e_i = \frac{b}{3} \begin{pmatrix} 0.7 \\ 2 \\ 0.3 \end{pmatrix}, \tag{20}$$

which are clearly different from (15) and (16) with $h^e = 1$.

Remarks

1. Despite the differences, the isoparametric concept still leads to a valid and convergent finite element solution. Further, in multi-dimensional problems it permits one to generate shape functions that can be used on general geometries. Note that the Lagrange shape functions have critical failures in two- and three-dimensions for general shaped elements – the isoparametric concept is needed.