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Structural Engineering,
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Consistency of Finite Difference Methods

Consider an equation of the form $\dot{y} = f(y, t)$ and a finite difference method applied to it. The discretization error is defined to be what you get when you take the finite difference method applied to the equation of interest and replace all the y_k values with their exact counter parts $y(t_k)$ (for any k).

1. First write down the finite difference method applied to the equation of interest. If we assume Forward Euler we have

$$\frac{y_{n+1} - y_n}{\Delta t} - f(y_n, t_n) = 0 \tag{1}$$

2. Now substitute in the exact solution (the result will not necessarily be zero now, so we call it τ).

$$\frac{y(t_{n+1}) - y(t_n)}{\Delta t} - \underbrace{f(y(t_n), t_n)}_{=\dot{y}(t_n)} = \tau \tag{2}$$

3. Manipulate to find how τ depends on Δt. In this case replace the exact solution at time t_{n+1} with a two term Taylor series with remainder: $y(t_{n+1}) = y(t_n) + \dot{y}(t_n)\Delta t + O(\Delta t^2)$. This results in

$$\tau(\Delta t) = O(\Delta t) \tag{3}$$

Thus one way of thinking about the discretization error, τ, is that it tells you the error of the equation you solve relative to the exact equation (it is the exact equation because you are using the exact $y(t)$).

Test your understanding: Here is another scheme that you can use to test your understanding, it is called the Leap-Frog method (and uses two prior time values, y_{n-1} and y_n, to move forward to the next time, y_{n+1}):

$$\frac{y_{n+1} - y_{n-1}}{2\Delta t} - f(y_n, t_n) = 0 \tag{4}$$

(a) Show that the discretization error $\tau(\Delta t) = O(\Delta t^2)$; i.e. the method is second order accurate (better than the Forward or Backward Euler methods). (b) Show that the method is *unconditionally unstable*! It never will work. It is always unstable. [To simplify the algebra in part (b), you can restrict your attention to the case where $\text{Im}(\lambda) = 0$ in the test equation.]