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**Notes on Error Estimates**

In class and in the class readers the error estimation results a not mathematically very precise. The results given are correct but the derivations are not precise and rely on heuristics and hand-waving in key locations. In these short notes, more precise error estimates are derived for a canonical problem. The results are given for a one-dimensional problem but can be extended to multiple dimensions with some extra effort.

# 1 Canonical Problem

Our canonical problem will be: Find $u(x)$ such that

$$u'' + f = 0 \tag{1}$$

over the interval $(0, 1)$ with boundary conditions $u(0) = u(1) = 0$, where $f(x)$ is known. The weak form expression of this problem is: Find $u \in \mathcal{S}$ such that

$$\int_0^1 v'u'\,dx = \int_0^1 vf\,dx \tag{2}$$

for all $v \in \mathcal{V}$. For this problem the appropriate trial solution and test spaces are the same due to the Dirichlet boundary conditions. If we first define the space $L^2 = \{\phi(x) \mid \int_0^1 \phi^2\,dx < \infty\}$, then they are appropriately defined by $\mathcal{S} = \mathcal{V} = \{\phi \mid \phi \in L^2,\ \phi' \in L^2 \text{ and } \phi(1) = \phi(0) = 0\}$. The space $L^2$ is known as the space of square integrable functions. Thus $\mathcal{S}$ and $\mathcal{V}$ consist of functions that are square integrable and whose first derivatives are square integrable. This space of functions is known as the Sobolev space $H^1$. The boundary condition restriction actually makes $\mathcal{S}$ and $\mathcal{V}$ a subset of $H^1$ known as $H^1_o$.

## 1.1 Inner products and norms

Before trying to look at finite element errors, we first need a few mathematical results about the our problem. Let us first introduce the $L^2$ inner product for functions $f, g \in L^2$, by

$$\langle f, g\rangle = \int_0^1 fg\,dx \tag{3}$$

and the natural norm

$$\|f\|_{L^2} = \sqrt{\langle f, f\rangle}\,. \tag{4}$$

Because we will be performing bounding estimates later, it is useful to note that these definitions obey the Cauchy-Schwarz inequality

$$|\langle f, g \rangle| \leq \|f\|_{L^2} \|g\|_{L^2} \,, \tag{5}$$

where $f, g \in L^2$.

Even though $H^1 \subset L^2$, we will also want a separate norm for functions in $H^1$. We will define this norm for $f \in H^1$ as

$$\|f\|_{H^1} = \sqrt{\|f\|_{L^2}^2 + \|f'\|_{L^2}^2} \,. \tag{6}$$

Note that this norm measures two things: the magnitude of the function itself as well as the magnitude of its derivative. The $L^2$-norm only measures the magnitude of the function. If we want to just measure the $L^2$ norm of the derivative, then we have what is called the $H^1$ semi-norm

$$|f|_{H^1} = \|f'\|_{L^2} \,. \tag{7}$$

In a similar fashion we can also define the $H^2$ semi-norm as

$$|f|_{H^2} = \|f''\|_{L^2} \,. \tag{8}$$

This norm measures the second derivative of a function. The reason these norms are called semi-norms is that a *true* norm is only zero when its input is zero. The semi-norms can give zero values for non-zero inputs but in all other aspects they behave (mathematically) as norms.

Lastly, let us define the $a$-form

$$a(f, g) = \int_0^1 f' g' \, dx \,. \tag{9}$$

With this last definition, the weak form expression reads $a(v, u) = \langle v, f \rangle$. Also notice that the $a$-form is a type of inner product; in our canonical example it gives the $L^2$ inner product of the derivatives of its arguments. If $f = g$ then it is just the square of the $H^1$ semi-norm of $f$. In mechanical problems $a(u, u)$ is twice the strain energy in the solution.

## 1.2 Properties of the canonical problem

With our new notation and definitions, let us note the following important mathematical properties of our problem:

1. The $a$-form is symmetric; i.e. $a(u, v) = a(v, u)$.

2. The $a$-form is bilinear; i.e. $a(\alpha u + \beta w, v) = \alpha a(u, v) + \beta a(w, v)$ for $\alpha, \beta \in \mathbb{R}$. By symmetry the same result holds in the second argument as well.

3. The $a$-form obeys the bounding property:

$$|a(f,g)| \le \|f'\|_{L^2}\|g'\|_{L^2} \le \gamma\|f\|_{H^1}\|g\|_{H^1}\,. \tag{10}$$

   The first inequality follows from the Cauchy-Schwarz inequality and the second from the fact that the $H^1$ norm of a function is always greater than or equal to its $L^2$ norm. This bounding property is sometimes referred to as a continuity estimate. The constant $\gamma$ can be taken to be trivially 1 in our case; in more general settings it can differ from unity but is always independent of $f$ and $g$.

4. The $a$-form also has an important bound from below which is known as an ellipticity or coercivity estimate. It says that

$$\alpha\|f\|^2_{H^1} \le a(f,f) \tag{11}$$

   for some constant $\alpha$ independent of $f$. This estimate is not too hard to prove for our canonical problem (with $\alpha = \frac{1}{2}$). In high dimensions, such estimates can be non-trivial to prove.

5. Lastly, note that the loading part of our problem satisfies the following estimate:

$$|\langle f,v\rangle| \le \|f\|_{L^2}\|v\|_{L^2} \le \Lambda\|v\|_{H^1}\,, \tag{12}$$

   where $\Lambda$ is a constant that essentially measures the severity of the load but is otherwise independent of $v$. This is a type of continuity estimate.

### 1.3 Stability of the exact solution

Before turning our attention to finite element estimates, let us first take note of one property of the exact solution. This property is known as a measure of the stability of the solution. Since $a(u,v) = \langle f,v\rangle$ for all $v \in \mathcal{V}$ and $\mathcal{S} = \mathcal{V}$, we can choose $v = u$ (the exact solution) and write

$$\alpha\|u\|^2_{H^1} \le |a(u,u)| = |\langle f,u\rangle| \le \Lambda\|u\|_{H^1}\,. \tag{13}$$

We can now divide both sides by the $H^1$ norm of the solution to give the estimate:

$$\|u\|_{H^1} \le \frac{\Lambda}{\alpha}\,. \tag{14}$$

This stability estimate tells us that magnitude of the solution (and its derivative) is bounded by the severity of the load and inversely by the coercivity constant.

## 2 Finite element error estimates

In the finite element method, the weak form equation is solved utilizing subspaces $\mathcal{S}^h \subset \mathcal{S}$ and $\mathcal{V}^h \subset \mathcal{V}$. This produces a solution $u^h \in \mathcal{S}^h$ for which

$$a(u^h,v^h) = \langle v^h,f\rangle \tag{15}$$

for all $v^h \in \mathcal{V}^h$.

## 2.1 Stability of the FEA solution

The finite element solution satisfies (15). If we use the exact same reasoning as we did for the exact solution, we see that the finite element solution satisfies the same stability estimate as the exact solution; viz.,

$$\|u^h\|_{H^1} \leq \frac{\Lambda}{\alpha} \,. \tag{16}$$

## 2.2 Orthogonality

As a first finite element estimate let us derive an orthogonality estimate. Note that since any $v^h \in \mathcal{V}^h$ is also in $\mathcal{V}$, we also have that

$$a(u, v^h) = \langle v^h, f \rangle \,, \tag{17}$$

where $u$ is the exact solution. Let us now define the error in the finite element solution as $e = u - u^h$. Then subtracting (15) from (17) and making use of the bilinearity property, we have that

$$a(e, v^h) = 0 \tag{18}$$

for all $v^h \in \mathcal{V}^h$. This tells us that the error in the finite element solution is orthogonal to space of functions $\mathcal{V}^h$ when measured in the $a$ inner product.

## 2.3 Best approximation: energy norm

Consider now any arbitrary element $U^h \in \mathcal{S}^h$. We can define error associated with this arbitrary trial solution as $E = u - U^h$. Note that $u^h$ and $U^h$ are both elements of $\mathcal{S}^h$ and thus differ from each other by an element of $\mathcal{V}^h$; i.e. $u^h - U^h = w^h$ for some $w^h \in \mathcal{V}^h$. This implies $E = u - (u^h - w^h) = e + w^h$. If we measure the "energy" of this error we see that

$$a(E, E) = a(e + w^h, e + w^h) = a(e, e) + 2\,\underbrace{a(e, w^h)}_{0} + \underbrace{a(w^h, w^h)}_{\geq 0} \,. \tag{19}$$

In this expansion, we have made use of the bilinearity and symmetry properties of the $a$-form. Now note that the middle term is zero by the orthogonality property of the FEA error to $\mathcal{V}^h$ and the last term is always non-negative. Thus we have the so-called *best approximation property*

$$a(e, e) \leq a(E, E) \,, \tag{20}$$

where $E$ represent the error associated with any other member of the space of trial solutions.

## 2.4 FEA error stability estimate

The error estimate derived in Section 2.3, tells us about the magnitude of the derivative of the FEA error relative to other possible choices in $\mathcal{S}^h$; it does not say anything about the error itself. To make such a statement, the coercivity estimate for the $a$-form can help us. We know from coercivity that,

$$\alpha\|e\|_{H^1}^2 \leq a(e,e) \underbrace{=}_{\text{orthogonality}} a(e, e+w^h) = a(e,E)\,, \tag{21}$$

where the first equality follows by orthogonality (18) for any $w^h \in \mathcal{V}^h$ and the second for the same reasoning as in Section 2.3. By continuity (10) we can bound $a(e,E) \leq \gamma\|e\|_{H^1}\|E\|_{H^1}$. Putting this together with (21), we have an $H^1$ error estimate which say

$$\|e\|_{H^1} \leq \frac{\gamma}{\alpha}\|E\|_{H^1}\,. \tag{22}$$

This error estimate gives a statement of how good the FEA error is relative to the error associated with any other trial solution in $\mathcal{S}^h$. Due to the appearance of the coercivity constant, this estimate, while being a type of best approximation estimate, is also a type of stability estimate.

## 2.5 Interpolation errors

Let us now consider more specifically finite element subspaces consisting of linear hat functions. If we use linear hat functions, the approximation space is linearly complete over individual elements. Due to the linear completeness of $\mathcal{S}^h$ over individual elements, we can choose a $U^h \in \mathcal{S}^h$ such that it exactly matches the true solution at the nodes. This function is known at the interpolant. Its error,

$$E = u - U^h\,, \tag{23}$$

is zero at $x_1^e$ and $x_2^e$ and thus passes through a stationary point for some $\xi \in (x_1^e, x_2^e)$. The exact location of $\xi$ is unknown. This tells us that

$$E'(x) = E'(x) - \underbrace{E'(\xi)}_{=0} = \int_\xi^x E''\,dx\,. \tag{24}$$

Note that $U^{h''} = 0$ since our appoximation space is linear over the element. This implies that the integral above can be replace by an integral solely in terms of the exact solution, yielding

$$E' \leq \int_\xi^x u''\,dx\,. \tag{25}$$

We can now take absolute values and exploit the Cauchy-Schwarz inequality to show

$$|E'| \leq \int_\xi^x |u''|\,dx \leq \int_{\Omega_e} |u''|\,dx \underbrace{\leq}_{\text{Cauchy−Schwarz}} \|1\|_{L^2(\Omega_e)}\|u''\|_{L^2(\Omega_e)} = h^{1/2}|u|_{H^2(\Omega_e)}\,, \tag{26}$$

where we have taken advantage of the fact that $\|1\|_{L^2(\Omega_e)} = \sqrt{\int_{\Omega_e} 1^2\,dx} = h^{1/2}$ and the definition of the $H^2$ semi-norm. If we now square both sides and integrate over the element we have:

$$|E|^2_{H^1(\Omega_e)} \leq h^2 |u|^2_{H^2(\Omega_e)}\,. \tag{27}$$

To convert this error bound over a single element to one over the entire domain, we can simply sum over the elements to give

$$|E|^2_{H^1} = \sum_{e=1}^{N} |E|^2_{H^1(\Omega_e)} \leq \sum_{e=1}^{N} h^2 |u|^2_{H^2(\Omega_e)} = h^2 \sum_{e=1}^{N} |u|^2_{H^2(\Omega_e)} = h^2 |u|^2_{H^2}\,. \tag{28}$$

Taking square-roots then provides us with an interpolation estimate or interpolation error bound in the $H^1$ semi-norm:

$$|E|_{H^1} \leq h |u|_{H^2}\,. \tag{29}$$

This last error estimate in on the first derivative of the error associated with the interpolant. We can similarly derive an estimate for the error of the interpolant by noting that at $x_1^e$ and at $x_2^e$ the interpolant error is zero. This implies:

$$E(x) = E(x) - \underbrace{E(x_1^e)}_{=0} = \int_{x_1^e}^{x} E'\,dx\,. \tag{30}$$

We can now proceed in an identical fashion as we did for our $H^1$ semi-norm estimate. This then gives that:

$$\|E\|^2_{L^2} \leq h^2 |E|^2_{H^1}\,. \tag{31}$$

If we now use (29), we can bound this error as

$$\|E\|^2_{L^2} \leq h^4 |u|^2_{H^2}\,. \tag{32}$$

Thus for the interpolant, $U^h \in \mathcal{S}^h$, we have that the error in the $L^2$ norm is bounded as

$$\|E\|_{L^2} \leq h^2 |u|_{H^2}\,. \tag{33}$$

Taken together, the $L^2$ norm interpolation estimate (33) and the $H^1$ semi-norm interpolation estimate (29) yield an $H^1$ norm interpolation estimate

$$\|E\|_{H^1} \leq h |u|_{H^2}\,. \tag{34}$$

## 2.6 An $L^2$ estimate on the FEA error

We are finally in a position to compute an error estimate for the FEA solution which shows how the error depends upon the discretization and the character of the exact solution. To do this we will use a so-called duality argument. The duality argument utilizes an auxiliary dual problem:

$$\phi'' + e = 0 \tag{35}$$

over the interval $(0, 1)$ with boundary conditions $\phi(0) = \phi(1) = 0$. Thus the dual problem is quite similar to our original problem except that the unknown field $\phi$ is "driven" by the finite element error $e = u - u^h$. There are some important observations that one should make about the dual problem:

1. The space of test functions and trial solutions for the dual problem are the same as for our actual problem.
2. The FEA error $e$ is an element of the space of test functions for the dual problem, $e \in \mathcal{V}$.
3. The norms of the dual field, $\phi$, and the FEA error are related as $|\phi|_{H^2} = \|e\|_{L^2}$. This property is one of the main reason for introducing the dual problem.
4. The weak form for the dual problem is: Find $\phi \in \mathcal{S}$ such that $a(\phi, v) = \langle e, v\rangle$ for all test functions $v \in \mathcal{V}$.
5. There exists an element $\Phi^h \in \mathcal{S}^h$ (and consequently also in $\mathcal{V}^h$) such that the interpolation estimates of Section 2.5 hold (relative to the dual problem). $\Phi^h$ is the interpolant of the exact solution, $\phi$, to the dual problem.

Using point 2 together with point 4, note first that

$$\|e\|^2_{L^2} = a(\phi, e)\,. \tag{36}$$

By orthogonality of the finite element error $e$ to the space of test functions $\mathcal{V}^h$, we then have

$$\|e\|^2_{L^2} = a(\phi, e) = a(\phi - \Phi^h, e)\,. \tag{37}$$

We can now use our continuity bound (10) on the last term on the left to give the estimate

$$\|e\|^2_{L^2} \le \gamma\|e\|_{H^1}\|\phi - \Phi^h\|_{H^1}\,. \tag{38}$$

We can now use our $H^1$ interpolation error estimate (34) applied to the dual problem to show that

$$\|e\|^2_{L^2} \le \gamma h\|e\|_{H^1}|\phi|_{H^2} = \gamma h\|e\|_{H^1}\|e\|_{L^2}\,, \tag{39}$$

where we have employed point 3 in writing the last equality on the right. Dividing through by the $L^2$ norm of the error we have

$$\|e\|_{L^2} \le \gamma h\|e\|_{H^1}\,. \tag{40}$$

We can now apply our finite element error stability estimate (22) together with the $H^1$ interpolation estimate (34) (for the original problem) to the last term on the right. This yields:

$$\|e\|_{L^2} \le Ch^2|u|_{H^2}\,, \tag{41}$$

Where we have lumped all the constants, which are independent of the exact solution and element size, into $C$.

Remarks:

1. The error is seen to decrease quadratically with respect to element size $h$ and the error is also seen to depend on the exact solution through the $H^2$ semi-norm; i.e. through a measure of the second derivative of the exact solution.

2. Thus to reduce error we need to reduce element size and pay attention to “strong” variations in the exact solution.

3. This is the same result that is claimed to be true in the course notes. For other problems and in higher dimensions, the details of the error analysis follows the exact same line but the details can be more involved.

4. The general developments can also be followed to obtain error estimates in other norms.