UNIVERSITY OF CALIFORNIA BERKELEY
Department of Civil Engineering
Spring 2019

Structural Engineering,
Mechanics and Materials
Professor: S. Govindjee

HW 9: Due Thursday May 2
Estimation of flow field pressures from measured velocity fields

Consider the incompressible Navier-Stokes equations, which govern the motion of an , incompressible Newtonian fluid

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \boldsymbol{v}) \cdot \boldsymbol{v} = \nu \nabla^2 \boldsymbol{v} - \nabla p \tag{1}$$

$$\operatorname{div} \boldsymbol{v} = 0\,, \tag{2}$$

where $\boldsymbol{v}$ is the fluid velocity, p is fluid pressure (scaled by the fluid density), and ν is the kinematic viscosity.

Assume that $\boldsymbol{v}$ is known. If we want to solve for pressure, we can define $\boldsymbol{b} = \nu \nabla^2 \boldsymbol{v} - \partial \boldsymbol{v}/\partial t - (\nabla \boldsymbol{v}) \cdot \boldsymbol{v}$ and take the divergence of (1) to get

$$\operatorname{div} [\nabla p - \boldsymbol{b}] = 0\,. \tag{3}$$

Applying the standard Galerkin procedure and integrating by parts gives

$$\int_\Omega \nabla q \cdot \nabla p \, dv = \int_\Omega \nabla q \cdot \boldsymbol{b} \, dv + \int_{\partial\Omega} q(\nabla p - \boldsymbol{b}) \cdot \boldsymbol{n} \, da\,. \tag{4}$$

Note, however, that the second term on the right-hand side goes to zero due to (1). Thus, the common finite element "do nothing" boundary condition (zero Neumann values) actually universally satisfies the normal component of the governing equation at the boundary. We are left with

$$\int_\Omega \nabla q \cdot \nabla p \, dv = \int_\Omega \nabla q \cdot \boldsymbol{b} \, dv\,. \tag{5}$$

This is the weak form for the consistent pressure Poisson equation (CPPE).

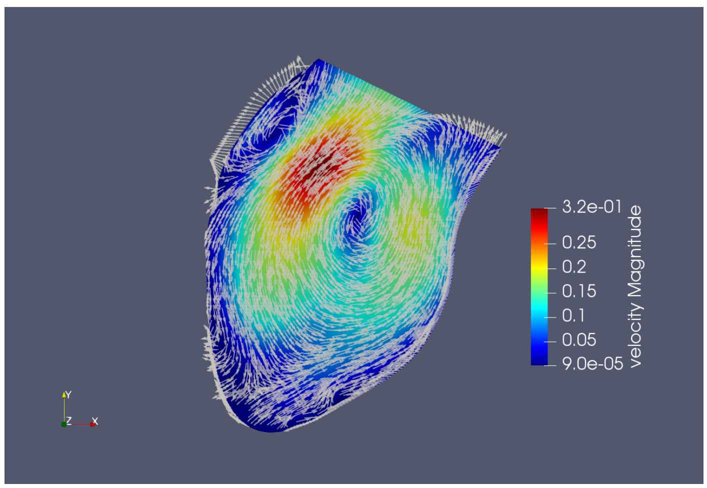
Another approach is to recognize that $\operatorname{div}[\partial \boldsymbol{v}/\partial t] = \partial \operatorname{div}[\boldsymbol{v}]/\partial t = 0$ and $\operatorname{div}[\nu \nabla^2 \boldsymbol{v}] = \nu \nabla^2 (\operatorname{div}[\boldsymbol{v}]) = 0$ by (2). In this case, application of the standard Galerkin procedure leads to

$$\int_\Omega \nabla q \cdot \nabla p \, dv = \int_\Omega q \operatorname{div} [(\nabla \boldsymbol{v}) \cdot \boldsymbol{v}] \; dv + \int_{\partial\Omega} q \nabla p \cdot \boldsymbol{n} \, da\,. \tag{6}$$

This weak form for the pressure Poisson equation may seem desirable since it does not require an evaluation of the Laplacian of velocity (which is embedded inside $\boldsymbol{b}$) or an approximation of the time derivative of velocity (also embedded inside $\boldsymbol{b}$). A trade-off, however, with (6) is that we are required to impose boundary conditions on p, which are generally unknown (in the setting of this problem). In this assignment we will use (5).

Note that (5) is the weak form for a (singular) Poisson problem due to the lack of any Dirichlet boundary conditions – meaning the solution is only unique to an additive constant. There are several techniques for solving these types of problems, including through the addition of a constraint on e.g. the norm of the solution, or less desirably (but expediently) specifying a Dirichlet boundary condition at an arbitrary boundary node (which is what we will do).

Application: The PPE can be very useful in scenarios where we can measure the velocity field $\boldsymbol{v}$ and want to find the pressure field p. In this case, (5) (or (6)) is nothing more than Poisson's equation since the right hand side is a known forcing function. A highly relevant application is the possibility of measuring the blood flow velocity inside the human heart using medical imaging, and then using that flow field to compute the pressure distribution inside the heart to aid diagnosis of cardiac disease. We will use two-dimensional velocity data inside the left ventricle (the primary pumping chamber) of a diseased heart. The velocity data spans the entire cardiac cycle. The velocity data is uploaded on Piazza, and the figure below shows a snapshot of the velocity field.



Assignment:

1. Derive eqns (4)-(6). Additionally write out a description of a FEA procedure for solving (5).

2. Use FEniCS to solve the CPPE, (5), using the standard Galerkin method. Much of the code has been written for you in `hw9_template.py`, uploaded on Piazza. Assume $\nu = 3.77 \times 10^{-6}\,\mathrm{m}^2/\mathrm{s}$.

3. Solve for the pressure and represent the solution as a color/contour plot. Display several snapshots throughout time (e.g. 6 instances of your choosing).

4. Compare the pressure plots with the velocity field plots. Discuss similarities in velocity and pressure fields based on physical principles.

Deliverables (100pts total):

1. Thoroughly yet concisely describe the problem, how you solved it, and the results and interpretations. This can be typeset or a combination of very neat handwriting and computer-generated plots bundled in the order below.
    (a) Introduction: Briefly describe the problem and goals (10pts)
    (b) Method: Describe your solution strategy and algorithm (40pts)
    (c) Results: Figures (40pts)
    (d) Discussion: Interpretation of the results (10pts)
2. Your Python code.

All in all you will submit a PDF for item 1 and Python file for item 2.