UNIVERSITY OF CALIFORNIA BERKELEY
Department of Civil Engineering
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Structural Engineering,
Mechanics and Materials
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HW 6: Due Thursday March 14

1. (20pts) Consider a 2D plane-strain 4-node quadrilateral element with nodes

$$\begin{aligned} \boldsymbol{x}_1^e &= (0,0) \text{ m} && (1)\\ \boldsymbol{x}_2^e &= (2,0) \text{ m} && (2)\\ \boldsymbol{x}_3^e &= (2,1) \text{ m} && (3)\\ \boldsymbol{x}_4^e &= (0,1) \text{ m} && (4)\\ & && (5) \end{aligned}$$

and nodal displacements

$$\begin{aligned} \boldsymbol{\Delta}_1^e &= (0,0) \text{ m} && (6)\\ \boldsymbol{\Delta}_2^e &= (0.001,0) \text{ m} && (7)\\ \boldsymbol{\Delta}_3^e &= (0.001,0.002) \text{ m} && (8)\\ \boldsymbol{\Delta}_4^e &= (0.0,-0.001) \text{ m} && (9)\\ & && (10) \end{aligned}$$

   (a) Find the strain field in the element.
   (b) Assume Lamé parameters $\lambda = 100$ GPa, $\mu = 50$ GPa, so that

$$\boldsymbol{D} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}.$$

   Find the stress field in the element.
   (c) Find the traction field on the edges of the element.
   (d) Find the (consistent) nodal loads at the three nodes of the element.

2. (20 pts) Consider a 2D domain $\Omega = \{(x,y) \mid (x,y) \in [0,L] \times [0,L]\}$, where $L = 2$ m, that has been uniformly discretized into a 10×10 mesh of bi-quadratic elements (9-node quadrilateral elements). Ω is subject to an edge traction $\bar{\boldsymbol{t}} = \sigma \frac{x}{L} \boldsymbol{e}_y$ on the top edge ($y = L$); assume $\sigma = 2$ MPa.

(a) Find the (consistent) nodal loads on the 21 nodes along the top edge. [Hint: Write a computer program to compute the values.]

(b) Plot these values as force vectors at the corresponding nodes. On the same plot, plot the traction distribution.

(c) Verify that the sum of the consistent nodal forces is equal to the integral of the traction field over the edge.

3. (20 pts) Consider the case of axis-symmetric elasticity, where no field depends on the polar angle θ and $u_\theta = 0$. This results in a situation where the problem becomes two-dimensional (just as in the practice midterm exam) and the primary unknowns are $u_r(r,z)$, $u_z(r,z)$. Note that the general relations for the strains in cylindrical coordinates are:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \tag{11}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{u_r}{r} \tag{12}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{13}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) \tag{14}$$

$$\gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \tag{15}$$

$$\gamma_{z\theta} = \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \tag{16}$$

(a) This implies that the strain vector $\boldsymbol{\varepsilon}$ can be collapsed to a 4×1 vector (eliminating the 2 terms that will always be zero), which can be written as $\boldsymbol{\varepsilon} = \tilde{\nabla}\boldsymbol{u}$ where $\boldsymbol{u}$ is a 2×1 vector with components $u_r(r,z)$ and $u_z(r,z)$. What is the expression for $\tilde{\nabla}$?

(b) The stiffness matrix in this setting is

$$\boldsymbol{K} = \int \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\, r dr dz \tag{17}$$

where $\boldsymbol{B} = [\boldsymbol{B}_1, \boldsymbol{B}_2, \cdots, \boldsymbol{B}_n]$. What is the expression for $\boldsymbol{B}_A$?