UNIVERSITY OF CALIFORNIA BERKELEY
Department of Civil Engineering
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Structural Engineering,
Mechanics and Materials
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HW 2: Due Thursday Feb. 7

1. Consider the functional

$$I(\phi) = \int_0^1 \left[\frac{1}{2} (\phi')^2 + x^2 \phi \right] dx + 6\phi(1) .$$

Minimize this functional over the space of trial solutions $\mathcal{S} = \{\phi \mid \phi(0) = 3\}$. In other words, compute the necessary conditions for $I(\phi)$ to be minimized over $\mathcal{S}$ — these conditions will be the weak form associated with this minimization problem.

2. Consider the strong form problem for $\phi(x)$

$$\phi'' + x^2 = 0 \qquad \forall x \in (0, 1) ,$$

where $\phi(0) = \phi(1) = 0$.

(a) Formulate the weak form problem for the given strong form problem using *admissible* test functions.

(b) Assume one-dimensional approximation spaces of the form

$$\mathcal{S}^h = \left\{ \phi^h(x) \mid \phi^h(x) = a(x - \langle x - 0.5 \rangle) \right\}$$
$$\mathcal{V}^h = \left\{ w^h(x) \mid w^h(x) = c(x - \langle x - 0.5 \rangle) \right\} ,$$

where a and c are scalar parameters and the angle brackets signify Macaulay brackets, $\langle y \rangle = y$ if $y > 0$, else it equals 0. Determine a and plot your approximate solution $\phi^h(x)$ versus the exact solution. All plots need to be computer generated, have proper axes labels, etc., i.e. use Matlab, Python, etc. to make them.

3. A circular elastic rod of length L is fixed at its base and embedded in a large soft elastic block. The top of the rod is subjected to an imposed torque T_L. The weak form problem that governs the rotation of the rod is as follows: Find $\varphi(z) \in \mathcal{S} = \{\phi(z) \mid \phi(0) = 0\}$ such that

$$\int_0^L \left[\frac{dw}{dz} GJ \frac{d\varphi}{dz} + \beta w \varphi \right] dz = w(L) T_L$$

is satisfied for all $w \in \mathcal{V} = \{w(z) \mid w(0) = 0\}$. The variables appearing in this relation have the following meanings: L is the length of the rod, G is the shear modulus of the rod, J is the polar moment of inertia of the rod, T_L is the torque applied to the rod, and β is a measure of the rotational restraint that the soft elastic block imparts to the rod.

The minimization form for this problem reads: Minimize $I(\varphi)$ over the space of trial functions $\mathcal{S} = \{\phi(z) \mid \phi(0) = 0\}$, where

$$I(\varphi) = \frac{1}{2}\int_0^L \left[GJ\left(\frac{d\varphi}{dz}\right)^2 + \beta\varphi^2\right] dz - T_L\varphi(L)\,. \tag{1}$$

Using classical methods for solving ordinary differential equations with constant coefficients, the exact solution can be found from the strong form. The result is

$$\varphi(z) = \left(\frac{T_L\lambda}{GJ}\right)\frac{\sinh(z/\lambda)}{\cosh(L/\lambda)}\,,$$

where $\lambda = \sqrt{GJ/\beta}$.

Using the two-dimensional approximation space $\mathcal{S}^h = \{\varphi^h(z) \mid \varphi^h(z) = a_1\psi_1(z) + a_2\psi_2(z)\}$ find and plot the solution to the minimization form problem (1) in comparison to the exact solution. Assume $\psi_1(z) = z$ and that $\psi_2(z) = \frac{1}{3}z^3$. To keep everything manageable, assume that $T_L = GJ = \beta = L = 1$ (all in consistent units).

On a single graph plot $\varphi(z)$ and $\varphi^h(z)$ versus z. On a second graph, plot the error $(\varphi(z) - \varphi^h(z))$ versus z.