UNIVERSITY OF CALIFORNIA BERKELEY
Department of Civil Engineering
Spring 2016

Structural Engineering,
Mechanics and Materials
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HW 6: (210 points total)

This homework assignment is to be done as homework as well for labs 10 and 11. Answers need to show all work and include related computer code.

1. (50 pts) Consider a plane-strain 3-node triangular element with nodes

$$\boldsymbol{x}^e_1 = (0,0) \text{ m} \tag{1}$$
$$\boldsymbol{x}^e_2 = (2,0) \text{ m} \tag{2}$$
$$\boldsymbol{x}^e_3 = (0,1) \text{ m} \tag{3}$$

Lamé parameters $\lambda = 200$ GPa and $\mu = 100$ GPa and density $\rho = 6000$ kg/m^3. At time $t = 0$ the nodal displacements are

$$\boldsymbol{\Delta}^e_1 = (-10^{-3}, -10^{-3}) \text{ m} \tag{4}$$
$$\boldsymbol{\Delta}^e_2 = (10^{-3}, 0) \text{ m} \tag{5}$$
$$\boldsymbol{\Delta}^e_3 = (0, 10^{-3}) \text{ m} \tag{6}$$

and the nodal velocities are

$$\boldsymbol{v}^e_1 = (0,0) \text{ m/s} \tag{7}$$
$$\boldsymbol{v}^e_2 = (0,0) \text{ m/s} \tag{8}$$
$$\boldsymbol{v}^e_3 = (0,0) \text{ m/s} \tag{9}$$

and the nodal accelerations are

$$\boldsymbol{a}^e_1 = (0,0) \text{ m/s}^2 \tag{10}$$
$$\boldsymbol{a}^e_2 = (0,0) \text{ m/s}^2 \tag{11}$$
$$\boldsymbol{a}^e_3 = (0,0) \text{ m/s}^2 \tag{12}$$

Using Newmark's method with $\beta = 0$, $\gamma = 1/2$ find and plot the x-component of local node 1 over the interval $[0, 0.005]$ s. You should lump the mass matrix using the row sum technique. If you observe that $\sum_{A=1}^{3} N_A(x,y) = 1$, it is easier to compute the components of the lumped mass matrix. Make sure to pick a time step that is sufficiently small to be stable and one that provides accuracy in the result.

2. (80 pts) Consider a one-dimensional elastic bar whose dynamic behavior is governed by

$$AEu'' = A\rho\ddot{u} \tag{13}$$

Assume that the bar is fixed at $x = 0$ and is free at $x = L$. With these boundary conditions the exact natural frequencies of vibration are given by

$$\omega_j^{\text{exact}} = \sqrt{\frac{E}{\rho}}\frac{1}{L}\left(\frac{\pi}{2} + j\pi\right) \qquad j \in \{0, 1, 2, \ldots\} \tag{14}$$

Find and plot the error in the finite element approximation for ω_0 and ω_3 when using linear as well as quadratic elements. The relative unsigned error $|\omega_j^{\text{fea}} - \omega_j^{\text{exact}}|/\omega_j^{\text{exact}}$ should be plotted versus L/d, where d is the nodal spacing. Use log-log scaling. The range of L/d should be sufficient for the relative unsigned error to drop below 10^{-6} for both vibrational modes. All your data should appear on a single plot.

Assume $L = 1$ m, $A = 0.01$ m^2, $E = 200$ GPa, and $\rho = 5000$ kg/m^3.

[Hint: Always sort the output of MATLAB's eigensolvers. You should observe that linear elements converge more slowly than the quadratic elements and the higher modes also converge slower than the lower modes.]

3. (80 pts) Consider a one-dimensional elastic bar, length $L = 50\ \mu$m, cross-sectional area $A = 0.1\ \mu\text{m} \times 0.1\ \mu$m, density $\rho = 4127$ kg/m^3, and Young's modulus $E = 139$ GPa. Assume the bar is fixed at both ends and excited at $x_a = L/2 - d$ and $x_b = L/2 + d$, where $d = 5\ \mu$m, by forces $f_a(t) = h\sin(2\pi f_o t)$ and $f_b(t) = h\sin(2\pi f_o t)$, where $h = 1$ mN and $f_o = 53.83$ GHz.

   (a) Construct a reduced order model of the system based upon the 13 modes nearest to f_o. In other words find the 13 frequencies nearest f_o as well as their associated eigenvectors and assume the solution is of the form $\Delta(t) = \sum_{j=1}^{13} d_j(t)\hat{\Delta}_j$.

   (b) Using the reduced order model compute the response of the system over the time interval $[0, 20]$ ns. As output, plot $(u_c + u_d)/2$ over the time interval, where $x_c = d$ and $x_d = L - d$. Assume the positions and velocities of all material points are zero at time zero.

[Hint: (1) Generate your mesh such that you always have nodes at x_a, x_b, x_c, and x_d. (2) Try to keep your element sizes uniform. (3) The generalized eigenvalue problem computes the circular frequencies squared. (4) The response quantity is very very small – order 10^{-22} m. (5) Make sure your mesh is sufficiently resolved. (6) The solution for each modal amplitude can be easily evaluated analytically due to the sinusoidal form of the loading. (7) When using `eigs` in MATLAB you need to remember to mass orthogonalize the vectors since MATLAB does not.]