UNIVERSITY OF CALIFORNIA BERKELEY
Department of Civil Engineering
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Structural Engineering,
Mechanics and Materials
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HW 4: Due Thursday March 3

Spherical coordinates can be helpful with this assignment. To that end, please recall the follow definitions:

$$x = r\sin(\phi)\cos(\theta) \tag{1}$$

$$y = r\sin(\phi)\sin(\theta) \tag{2}$$

$$z = r\cos(\phi) \tag{3}$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \in [0,\infty) \tag{4}$$

$$\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \qquad \in [0,\pi] \tag{5}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \in [0,2\pi) \tag{6}$$

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi + \frac{1}{r\sin(\phi)}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta \tag{7}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2\sin^2(\phi)}\frac{\partial^2 f}{\partial \theta^2} \tag{8}$$

$$dV = dx\,dy\,dz = r^2\sin(\phi)\,dr\,d\phi\,d\theta \tag{9}$$

$$dA_r = r^2\sin(\phi)\,d\phi\,d\theta \tag{10}$$

1. (20 pts) Consider a spherical domain $\Omega = \{\boldsymbol{x} \mid \|\boldsymbol{x}\| < R\}$, where $R = 0.5$ m. Let the domain be composed of a ceramic with isotropic thermal conductivity $k = 0.19\ \mathrm{W/(m\cdot K)}$ and constant mass density $\rho = 3.5\ \mathrm{g/cm^3}$. Assume a temperature field $T(r,\phi,\theta) = T_o + T_d\frac{r}{R}$, where $T_o = 20$ C and $T_d = 100$ C.

   (a) Find and plot the heat flux field over the domain; note the temperature field is spherically symmetric, so it is sufficient to only plot the radial component of $\boldsymbol{q}$ as a function of r.

   (b) What is the total energy per unit time moving through the outer surface of the sphere? Is the energy moving in or out of the sphere?

   (c) Is this temperature field a steady state temperature distribution for a system with $r_{\mathrm{s}}(r,\phi,\theta) = 0$, where $r_{\mathrm{s}}(r,\phi,\theta)$ is the heat supply? If not, what must $r_{\mathrm{s}}(r,\phi,\theta)$ equal for the system to be in steady state? Specify and plot if not zero.

2. (50 pts) Consider a spherical domain $\Omega = \{\boldsymbol{x} \mid \|\boldsymbol{x}\| < R\}$, where $R = 0.5$ m. Let the domain be composed of a ceramic with isotropic thermal conductivity $k = 0.19$ W/(m · K) and constant mass density $\rho = 3.5$ g/cm^3. The radial heat flux at $r = 0$ is specified to be zero, $-\boldsymbol{q} \cdot \boldsymbol{e}_r = 0$ at $r = 0$. The temperature at $r = R$ is specified to be $T_R = 50$ C. Assuming the heat supply is non-uniform with $r_{\mathrm{s}}(r) = [5 \sin(3\pi r/R) + 5]$ W/kg, you are to find the temperature and heat flux inside the sphere using a finite element approximation. Note that the boundary conditions dictate that this problem is spherically symmetric; thus the temperature field is only a function of r and the only non-zero component of the heat flux vector is $q_r = \boldsymbol{q} \cdot \boldsymbol{e}_r$.

   (a) Define the space of trial solutions and the space of test functions.

   (b) Find the weak form. Hint: when you integrate over the volume, you can explicitly integrate with respect to ϕ and θ, leaving a weak form that only involves integration with respect to r; hence, the problem is actually one-dimensional. The integrands should look like the ones we have encountered already but will contain extra factors of r^2 and 4π.

   (c) Write a MATLAB program to solve for the temperature field and the heat flux field. You may use elements of your choosing but your final answer for both fields needs to include converged plots. [Hint: The temperature at the center of sphere should be roughly 4000 C and the normal heat flux at $r = R$ should be roughly $q_n = 3800$ W/m^2.]