CE 133 / ME 180, Lab project #5

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The following lab statement has two purposes: 1) test your code, so that you make sure it is correct; and 2) solve some problems using your FEM code.

1   Testing your code

It is good practice to check your code against known results. Note that, when you check answers, the problems tend to be easy because you require the exact solution as a check. For this reason, think of the following *toy problem*.

Assume a domain, with generated mesh, as shown in Figure 1. This domain is a square of side 1. Note that $nelx = nely = 2$, and take $k = 10$. From the boundary conditions defined in Lab 4, also take the left and bottom edges to have $\bar{T} = 0$, and $\bar{T} = 10x$ for the top edge (node 7 included). Take the imposed flux on the right edge as $\bar{\mathbf{q}} = 0$ and the heat supply as $r = 0$.
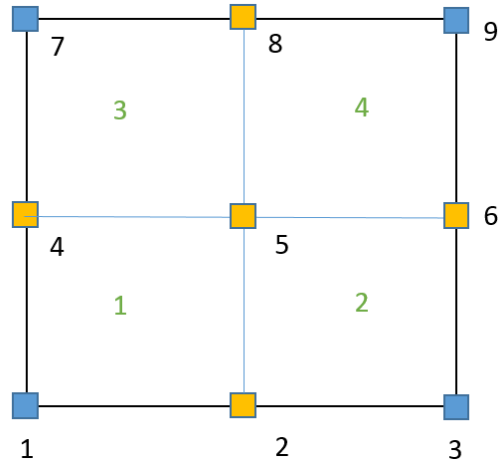

Figure 1: Mesh for the test problem.

First, check that your mesh generator provides the following information: $idd = (1, 2, 3, 4, 7, 8, 9)$ and $idq = (6)$, not necessarily in this order. Also, check your nodal coordinates (Table 1) and LM array (Table 2).

Node	x	y
1	0	0
2	0.5	0
3	1	0
4	0	0.5
5	0.5	0.5
6	1	0.5
7	0	1
8	0.5	1
9	1	1

Table 1: Coordinates of nodes, according to proposed enumeration.

1	2	4	5
2	3	5	6
5	6	8	9
4	5	7	8

Table 2: LM array, according to the proposed enumeration.

Once you know your mesh is correct, check the stiffness matrix for element 1. These are the exact values for the stiffness, which you should get by using a 2x2 quadrature rule.

$$
\mathbf{k}^1 = \begin{bmatrix}
6.67 & -1.67 & -3.33 & -1.67 \\
-1.67 & 6.67 & -1.67 & -3.33 \\
-3.33 & -1.67 & 6.67 & -1.67 \\
-1.67 & -3.33 & -1.67 & 6.67
\end{bmatrix}
$$

Are the stiffness matrices for the other elements the same as for this element? Why?

Note that the force vector is identically equal to zero in this case as $r = \|\bar{\mathbf{q}}\| = 0$.

For this check case, the global stiffness for the unknown degrees of freedom, $\mathbf{K}_{uu}$, is a 2x2 matrix:

$$
\mathbf{K}_{uu} = \begin{bmatrix}
4k_{33}^1 & 2k_{34}^2 \\
2k_{34}^2 & 2k_{33}^2
\end{bmatrix} = \begin{bmatrix}
26.67 & -3.33 \\
-3.33 & 13.33
\end{bmatrix}
$$

whereas the stiffness matrix $\mathbf{K}_{uk}$ is a 2x7 matrix, which multiplies the vector of known T's (nodes in idd). Can you say what are the components of this matrix? They will depend on the order of your idd array. Check your code in order to see if they are correct.

The solution to this problem can actually be computed analytically by using separation of variables. The exact solution of the problem is:

$$
T(x,y) = \sum_{n=1}^{\infty} \frac{2}{\sinh\left(\frac{(2n-1)\pi}{2}\right)} \left\{ \int_0^1 10z \sin\left(\frac{(2n-1)\pi z}{2}\right) dz \right\} \sin\left(\frac{(2n-1)\pi x}{2}\right) \sinh\left(\frac{(2n-1)\pi y}{2}\right)
$$

$$(1)$$

Use the former expression to check the two degrees of freedom of the model, i.e. the temperature at nodes 5 and 6. You should be able to obtain a decent result by just using 3 or 4 series terms.

2 Case 1. Solve the Laplace equation.

Now that you have a good understanding of your code and you have done some checks, it's time to see some more complex results. Note that the FEM code you've worked on is actually very powerful, and can solve the Laplace equation.

Let $\mathbf{x}_1$ to $\mathbf{x}_4$ be the four corners of a square of side 10, where $\mathbf{x}_1$ is the origin of the coordinate system and the square occupies the 1st quadrant. Denote by Ω the latter domain.

Solve the following boundary value problem:

$$
\begin{aligned}
\boldsymbol{\nabla}^2 u &= y + 1 \text{ in } \Omega \\
\bar{u} &= x \text{ on } \{y = 0\} \\
\bar{u} &= 5x \text{ on } \{y = 10\} \\
\bar{u} &= 0 \text{ on } \{x = 0\} \\
\boldsymbol{\nabla} u &= (5, 2) \text{ on } \{x = 10\}
\end{aligned}
$$

$$(2)$$

Use mesh sizes, in terms of $nelx \times nely$, of 2×2, 3×2, 2×3 , 5×5, 15×15 and 50×50. Provide:

1. A plot of the solution surface (Matlab function **surf**) for all the meshes.
2. A line plot of u and $\nabla u \cdot \mathbf{n}$ along $x = 10$, thus as a function of y (note that this is the edge where the flux is imposed). Does your plot of $\nabla u \cdot \mathbf{n}$ make sense? Think about what it should be.
3. For the 2×2 mesh, compute the flux on the edge 2-3 of element 1, and the flux on the edge 1-4 of element 2. Are they the same? Should they be the same? Repeat this for the 50×50 mesh (note that you want to compare the same edges, but now of elements 25 and 26) and comment on your findings.

Note: Fine meshes like 50×50 can take a long time when using symbolic functions. For this reason, you may prefer to perform your calculations up to 15×15 (substitute the 50×50 mesh for 15×15 mesh in part 3)).

3 Case 2. Solve the stationary heat equation.

Solve the stationary heat equation in the domain defined by the nodes $\mathbf{x}_1 = (0,0)$, $\mathbf{x}_2 = (5,0)$, $\mathbf{x}_3 = (1,8)$ and $\mathbf{x}_4 = (6,8)$. Take the density as $\rho = 2 \ g/cm^3$ and the conductivity as $k = 100 \ W/mK$. Assume the boundary conditions, where temperatures are in Kelvin:

$$
\begin{aligned}
\bar{T} &= 10 \text{ on } \{y = 0\} \\
\bar{T} &= 49 + x \text{ on } \{y = 8\} \\
\bar{T} &= 10 + 5y \text{ on } \{y - 8x = 0\} \\
-100\nabla T &= (2, -1) \text{ on } \{y - 8(x - 5) = 0\}
\end{aligned}
\tag{3}
$$

Finally, consider a heat supply of $r = \frac{y}{2} + 1 \ [W/g]$.

Use mesh sizes, in terms of $nelx \times nely$, of 2×2, 10×10, 50×50, 100×100 and 500×500. Consider the last mesh to be the exact result.

Provide:

1. A single plot with the temperature at the edge $\{y - 8(x - 5) = 0\}$ as a function of y for all your meshes.
2. A semilogx plot of the error at the center point of the domain as a function of the mesh size ($nelx$, for instance).

Note: Fine meshes like 100×100 or 500×500 can take a long time when using symbolic functions. For this reason, you may prefer to perform your calculations for 2×2, 10×10 and 20×20, taking the latter mesh as the exact result.