CE 133 / ME 180, Lab project #2

Ahmed Bakhaty, Miquel Crusells-Girona

Week #2 (2/1/16 - 2/7/16)

In this second lab, we will be studying the effects of mesh refinement, and how it effects the convergence of the FE solution.

To this purpose, we will consider the torsion of a cylindrical[1] bar. If one asumes that cross sections remain plane and there is no other force acting on the section than a torque, then:

$$\mathbf{u}(x, y, z) = (0, -z\phi, y\phi) \tag{1}$$

where ϕ is the twist of the cross section around its centroid.

With some manipulations and from the assumed displacement field, one is able to obtain[2] the equilibrium equation in differential form:

$$\frac{dT}{dx} + \bar{t} = 0 \tag{2}$$

where T represents the torque $T = \int_A (y\sigma_{xz} - z\sigma_{xy})\, dA$ and $\bar{t}$ is the external distributed torque per unit length applied to the bar.

If the material is linear elastic, and there is no warping, one can assume the constitutive model:

$$T = GJ\frac{d\phi}{dx} \tag{3}$$

where GJ represents the torsional stiffness of the section, G is the shear modulus and J is the polar moment of inertia.

As a result, the strong form of the homogeneous (non-warping) torsion problem is:

$$\begin{aligned} \frac{d}{dx}\left(GJ\frac{d\phi}{dx}\right) + \bar{t} = 0 \ &\text{ in } \Omega = (0, L) \\ \phi = \bar{\phi} \ &\text{ on } \Gamma_u = \{0\} \\ T = GJ\frac{d\phi}{dx} = \bar{T} \ &\text{ on } \Gamma_t = \{L\} \end{aligned} \tag{4}$$

Perform the following steps to analyze the solution to 4.

1 Solve for the exact solution

In order to assess the FEM solution of this problem, we will first solve for the exact solution. Consider the following data:

1. The radius of the section is R and the length of the bar is L, both constant.
2. $\bar{t} = A(L - x)$, with A constant.
3. J is the polar moment of inertia of the circle, and the shear modulus G is constant.
4. $\bar{\phi} = 0$ and $\bar{T}$ is a given non-zero constant.

[1] Note that a cylindrical bar does not warp. Also, if the section were not symmetric, one would have to consider the difference between the shear center and the centroid of the section.

[2] Calculate strains and use them into the internal virtual work.

2 Analysis of the FEM solution

Run a 1D FEM analysis of the non-warping torsion problem. To this purpose:

1. Consider: $R = 0.5$ m, $L = 10$ m, $G = 1$ GPa, $\bar{\phi} = 0$, $\bar{T} = 10$ kNm and $A = 1000$ kN/m.
2. Use a 1D problem, PDE Coefficient form, stationary analysis.
3. Create the geometry.
4. Use the appropriate boundary conditions.
5. Choose linear elements.

Run the model for element sizes: 1, 0.5, 0.25, 0.1, 0.05, 0.01 (m), and save a *txt* file of each set of results. Remember that this can be done by extracting *Data* from the report tab[3]. This procedure is called **h-refinement**.

For each set of solutions, compute[4] the error e as a function of position using the relation:

$$e = \frac{d_{FEM} - d_{exact}}{d_{exact}} * 100 \tag{5}$$

where d_{FEM} is your FEM value and d_{exact} is the exact value of the solution you computed in Section 1.

Submit, in the default format, the following answers:

- The exact solution in terms of R, L, G and $\bar{T}$ (development and final expression).
- Is this a linear problem? Can you check that with *Comsol*?
- For the cases $h = 1$ and $h = 0.1$, plot the finite element solution and the exact solution for ϕ and $grad\,\phi$ as a function of x. Physically explain the spatial variation that you see in both functions.
- Plot the error as a function of x for your 6 sets of solutions (in one plot). Include the code or excel spreadsheet you have used (in a clean and understandable format), printed to pdf.

[3] As you did in Lab 1.

[4] You can do this by using Excel, Matlab or any code you like.