UNIVERSITY OF CALIFORNIA BERKELEY | Engineering 7
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Linear Equations: Engineering Supplement

1 Introduction

The workhorse models of engineering are linear. There are two primary reasons for this: (1) linear response is the first approximation to any system's behavior and (2) linear equations are easy to solve and understand. In these supplementary notes we look at how such models (equations) arise in a few characteristic engineering settings. In the process we will also introduce a few important concepts that are *universal* across all engineering disciplines. Our examples will be drawing from electrical engineering, mechanical engineering, and civil engineering. In particular, we will look at the derivation of the governing equations for resistive circuits of electrical, thermal, and fluid systems. All these systems follow the same basic physical principles and share a common set of mathematical steps. While in these brief notes we can not teach you everything about these subjects, we can get you started in a meaningful way.

There are 3 basic principles that are applied to all these systems.

1. Conservation laws.
2. Compatibility conditions.
3. Constitutive relations.

The exact meanings of these three statements is best left to discussion within the context of particular physical problems.

2 Electrical Systems

In electrical resistive circuits the flowing substance is electrical charge commonly known as the current. At every node in an electrical circuit the sum of the currents at the node must equal zero; i.e. no electrical charge is allowed to accumulate at the node. This is principle 1 from above and is known as Kirchhoff's Current Law (KCL). The notion of compatibility says that the sum of the voltage drops around any closed loop in the system must add to zero. This is just another way of stating that the voltage at any point in the circuit has only one well defined value. This is known as Kirchhoff's Voltage Law (KVL). The constitutive relation is the equation that describes the behavior of electrical resistors – in this case Ohm's law.

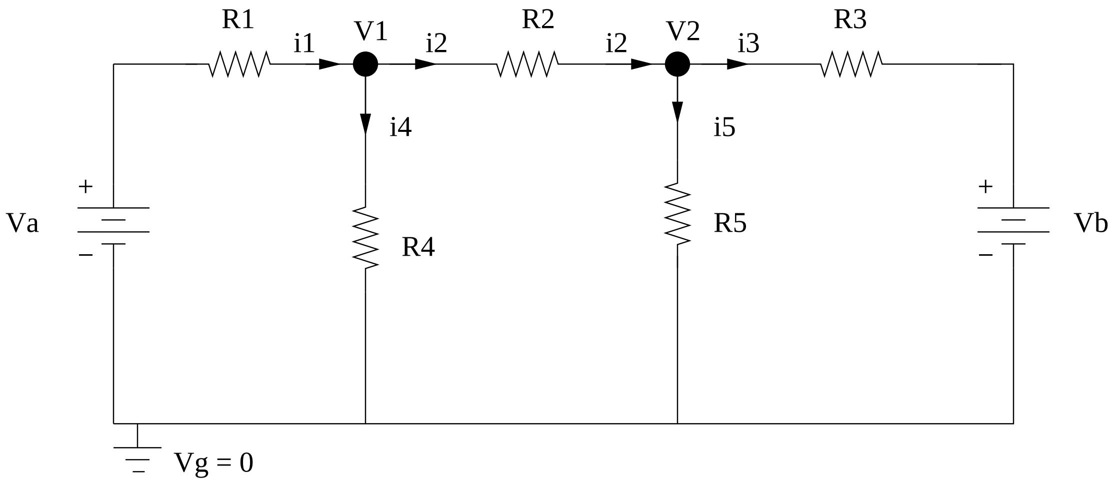

Figure 1: Example 1

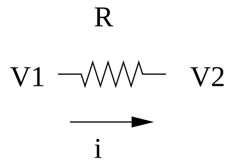

Figure 2: Ohm's Law

2.1 Example 1

Consider the circuit shown in Fig. 1. For given voltages V_a, V_b, and resistances R_1–R_5, find all the nodal voltages (V_1, V_2) and branch currents $(i_1, i_2, i_3, i_4, i_5)$ in the circuit.[1]

The KCL equations at the two "nodes" corresponding to the unknown voltages in the circuit are simply given as:

$$i_1 - i_2 - i_4 = 0 \tag{1}$$

$$i_2 - i_3 - i_5 = 0 \tag{2}$$

The constitutive relation for a resistor is that the voltage drop is given as $V_{\text{drop}} = Ri$ where R is a resistor's resistance and i is the current flowing through the resistor. The drop is given in the "direction" of i; see Fig. 2 where one would write $V_1 - V_2 = Ri$. For our circuit this gives:

$$\frac{1}{R_1}(V_a - V_1) - \frac{1}{R_2}(V_1 - V_2) - \frac{1}{R_4}(V_1 - 0) = 0 \tag{3}$$

$$\frac{1}{R_2}(V_1 - V_2) - \frac{1}{R_3}(V_2 - V_b) - \frac{1}{R_5}(V_2 - 0) = 0 \tag{4}$$

[1] A consistent set of units is voltage in volts, resistance in ohms, and current in amps.

We can now re-arrange these equations into a set of linear equations as follows:

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_4} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_5} \end{bmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V_a/R_1 \\ V_b/R_3 \end{pmatrix} \tag{5}$$

Within MATLAB these equations can be defined and solved using the following (assuming R1-R5 and Va, Vb are already defined):

```
>> b = [ Va/R1 ; Vb/R3];
>> A = [ 1/R1+1/R2+1/R4      -1/R2;
        -1/R2                1/R2+1/R3+1/R5];
>> x = A\b;
>> V1 = x(1);
>> V2 = x(2);
```

At this stage `x` represents the vector of unknown voltages. The currents in any resistor can now be obtained from the constitutive relations (since the voltages are all known).

2.2 Example 2

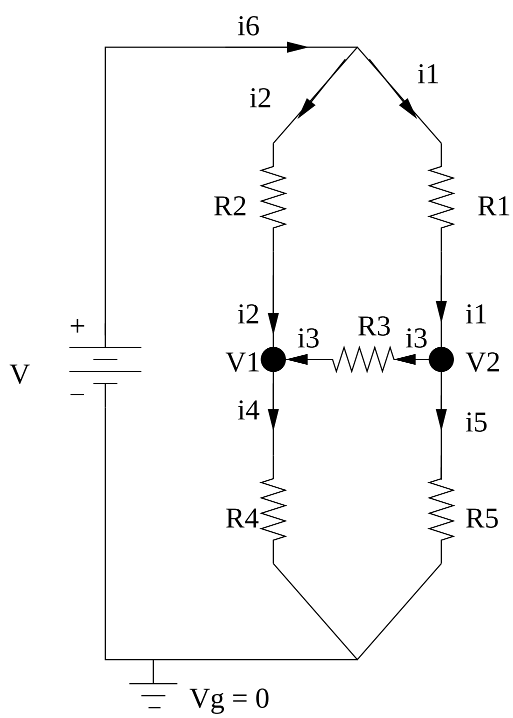

Figure 3: Example 2

As a second worked example of an electrical circuit consider the drawing in Fig. 3. Again the question is to find the nodal voltages (V_1, V_2) and the branch currents assuming that the voltage V and the resistances are all known. There are 2 unknowns in this problem.

The application of KCL to the 2 nodes with unknown voltages gives:

$$i_2 + i_3 - i_4 = 0 \tag{6}$$
$$i_1 - i_3 - i_5 = 0 \tag{7}$$

The application of the constitutive relations yields:

$$\frac{1}{R_2}(V - V_1) + \frac{1}{R_3}(V_2 - V_1) - \frac{1}{R_4}(V_1 - 0) = 0 \tag{8}$$
$$\frac{1}{R_1}(V - V_2) - \frac{1}{R_3}(V_2 - V_1) - \frac{1}{R_5}(V_2 - 0) = 0 \tag{9}$$

Converting to a system of linear equations in matrix form gives:

$$\begin{bmatrix} \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} & -\frac{1}{R_3} \\ -\frac{1}{R_3} & \frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_5} \end{bmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} V/R_2 \\ V/R_1 \end{pmatrix} \tag{10}$$

The MATLAB version of these equations with solution is given by

```
>> b = [ V/R2 ; V/R1];
>> A = [ 1/R2+1/R3+1/R4    -1/R3;
        -1/R3               1/R1+1/R3+1/R5];
>> x = A\b;
>> V1 = x(1);
>> V2 = x(2);
```

Again, currents can be obtained from the constitutive relations; e.g. `i2 = (V-V1)/R2` – note the first entry in the vector `x` corresponds to voltage V_1.

3 Fluid flow in pipes

Another engineering system that can be analyzed in much the same way is the flow of fluids in pipes. In such a situation, KCL amounts to the statement that the net mass flow into a node be zero. KVL amounts to the statement that the pressure is uniquely given at each point in the system. The constitutive relations give the pressure drop across each pipe in terms of the mass flux(flow) in the pipe and the pipe's properties.

3.1 Example 3

Consider the piping system shown in Fig. 4. For given pressures p_a and p_b we wish to find the nodal pressures $(p_1, \cdots, p_4)$ and the mass flux through the pipes $(q_1, \cdots, q_7)$. Each pipe

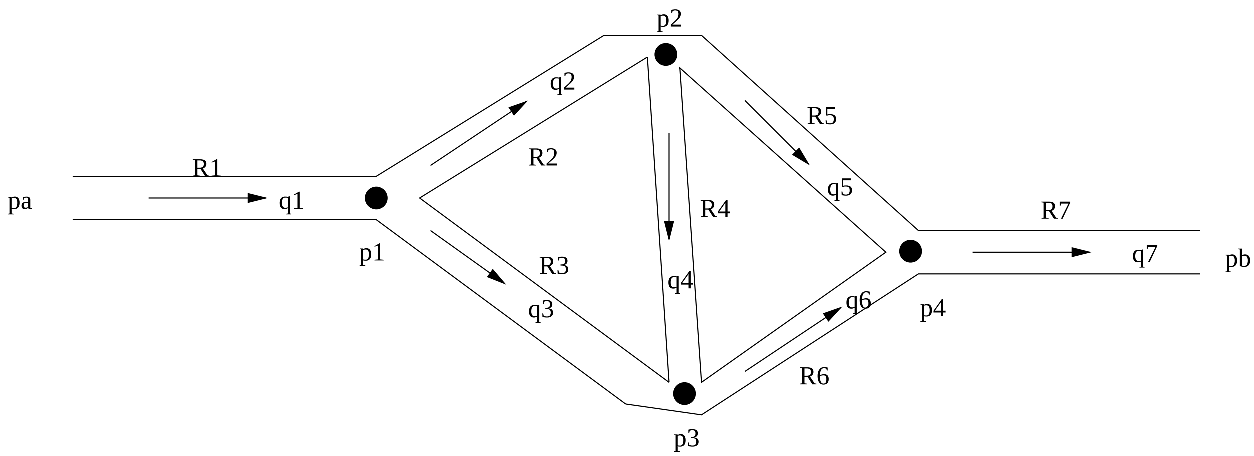

Figure 4: Example 3

is assumed to have a known resistance, R, to fluid flow in such a manner that the pressure drop from one end of the pipe to the other (in the direction of the flow) is given as $p_{\text{drop}} = Rq$ where q is the mass flux (flow) through the pipe.[2]

The KCL equations for the 4 "nodes" at the forks in the piping system give:

$$q_1 - q_2 - q_3 = 0 \tag{11}$$
$$q_2 - q_4 - q_5 = 0 \tag{12}$$
$$q_3 + q_4 - q_6 = 0 \tag{13}$$
$$q_6 + q_5 - q_7 = 0 \tag{14}$$

Thus, the mass flux entering each fork from one pipe must leave through the other 2 pipes. Applying the constitutive relations for each pipe, to convert fluxes to pressures, gives:

$$\frac{1}{R_1}(p_a - p_1) - \frac{1}{R_2}(p_1 - p_2) - \frac{1}{R_3}(p_1 - p_3) = 0 \tag{15}$$
$$\frac{1}{R_2}(p_1 - p_2) - \frac{1}{R_4}(p_2 - p_3) - \frac{1}{R_5}(p_2 - p_4) = 0 \tag{16}$$
$$\frac{1}{R_3}(p_1 - p_3) + \frac{1}{R_4}(p_2 - p_3) - \frac{1}{R_6}(p_3 - p_4) = 0 \tag{17}$$
$$\frac{1}{R_6}(p_3 - p_4) + \frac{1}{R_5}(p_2 - p_4) - \frac{1}{R_7}(p_4 - p_b) = 0 \tag{18}$$

[2] A consistent set of units for these quantities is: pressure in Pa, mass flux in kg/sec, and resistance in Pa sec/kg.

We can now re-arrange these equations into a set of 4 linear equations as follows:

$$\begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & -\frac{1}{R_3} & 0 \\ -\frac{1}{R_2} & \frac{1}{R_2}+\frac{1}{R_4}+\frac{1}{R_5} & -\frac{1}{R_4} & -\frac{1}{R_5} \\ -\frac{1}{R_3} & -\frac{1}{R_4} & \frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_6} & -\frac{1}{R_6} \\ 0 & -\frac{1}{R_5} & -\frac{1}{R_6} & \frac{1}{R_5}+\frac{1}{R_6}+\frac{1}{R_7} \end{bmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} p_a/R_1 \\ 0 \\ 0 \\ p_b/R_7 \end{pmatrix} \tag{19}$$

In MATLAB these would be input and solved as:

```
>> b = [ pa/R1 ; 0 ; 0; pb/R7 ];
>> A = [1/R1+1/R2+1/R3 −1/R2          −1/R3          0;
        −1/R2          1/R2+1/R4+1/R5 −1/R4          −1/R5;
        −1/R3          −1/R4          1/R3+1/R4+1/R6 −1/R6;
        0              −1/R5          −1/R6          1/R5+1/R6+1/R7];
>> x = A\b;
```

The vector `x` contains at this point the values of the nodal pressures in the piping system. Mass fluxes can be obtained from the constitutive relations.

4 Thermal systems

Thermal systems can also be analyzed in a similar manner. In such systems, temperature plays the role of voltage or pressure and heat flux (energy flux) plays the role of electric current or mass flux. KCL has the meaning of saying the net energy flowing into a point must sum to zero; i.e. energy can not be stored in the system. KVL in this setting means that every point in the system has a well defined temperature (i.e. is uniquely defined).

4.1 Example 4

Consider the thermal system shown in Fig. 5. If you want you can think of this as a very simple model of the heat transfer problem through the space shuttle wing. There is a layer of insulation that is exposed to a plasma at a given temperature T_p and the interior is hopefully kept at some nice (assumed known here) temperature T_i. In between the insulation and the aluminum frame is a layer of adhesive material. The unknowns here are the heat fluxes (h_1, h_2, h_3) through the layers and the junction temperature (T_{j1}, T_{j2}). The thermal resistances of the materials are assumed known.[3] The temperature drop (in the direction of the heat flux) across any given layer of material is given by $T_{\text{drop}} = Rh$, where R is the thermal resistance of the layer and h is the heat flux.

[3] A consistent set of units for these types of problems are temperature in K, heat flux in J/s, and thermal resistance in K s/J.

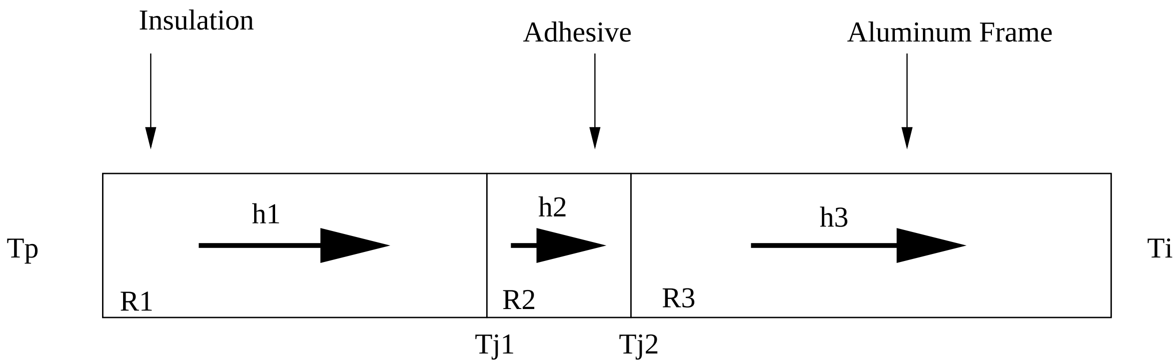

Figure 5: Example 4

If we apply KCL at the interfaces between the layers we get:

$$h_1 - h_2 = 0 \tag{20}$$
$$h_2 - h_3 = 0 \tag{21}$$

Incorporating the constitutive relations gives:

$$\frac{1}{R_1}(T_p - T_{j1}) - \frac{1}{R_2}(T_{j1} - T_{j2}) = 0 \tag{22}$$
$$\frac{1}{R_2}(T_{j1} - T_{j2}) - \frac{1}{R_3}(T_{j2} - T_i) = 0 \tag{23}$$

Re-writing in matrix form gives:

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_3} \end{bmatrix} \begin{pmatrix} T_{j1} \\ T_{j2} \end{pmatrix} = \begin{pmatrix} \frac{1}{R_1} T_p \\ \frac{1}{R_3} T_i \end{pmatrix} \tag{24}$$

The implementation in MATLAB should be self evident.