Error Estimate for Trapezoidal Rule

The integral of a function $f(x)$ over an interval (a, b) will be denoted as

$$I = \int_a^b f(x)\, dx\,. \tag{1}$$

The expression for the Trapezoidal rule approximation to the integral is given as

$$I_T = \frac{b-a}{2}(f(a) + f(b))\,. \tag{2}$$

Using Taylor's theorem with remainder one can show that

$$I - I_T = -\frac{f''(c)h^3}{12}\,, \tag{3}$$

where c is some (unknown) point between a and b and $h = b - a$. To be able to exploit this relation, we need to know $f''(c)$, which is not easy to know. Notwithstanding, we can estimate it in the same way as we did in our adaptive differentiation algorithm.

Consider computing the integral using a single panel of size h and then with two panels of size $h/2$. This will give two different approximations with the exact error relations

$$I - I_T(h) = -\frac{f''(c_1)h^3}{12} \tag{4}$$

$$I - I_T(h/2) = -\frac{f''(c_2)(h/2)^3}{12} - \frac{f''(c_3)(h/2)^3}{12}\,, \tag{5}$$

where $c_1 \in (a, b)$, $c_2 \in (a, a + h/2)$, and $c_3 \in (a + h/2, b)$ are all unknown. Note that by $I_T(h/2)$ we mean the sum of the application of the trapezoidal rule to TWO panels of size $h/2$. If we now subtract these two results and ASSUME that $f''(x)$ is approximately constant between a and b, then

$$I_T(h/2) - I_T(h) \approx -\frac{f''h^3}{12}\left[1 - \frac{1}{4}\right]\,. \tag{6}$$

Thus the error in $I_T(h)$ is approximately $\frac{4}{3}(I_T(h/2) - I_T(h))$ or, equivalently, the error in $I_T(h/2)$ is approximately $\frac{1}{3}(I_T(h/2) - I_T(h))$. This allows us to construct a simple function that computes the integral to a "known" error tolerance via a process of recursive divide and conquer:

1. Compute approximate integral values using one panel of size h and two panels of size $h/2$.

2. Estimate the absolute error as $|I_T(h/2) - I_T(h)|/3$

3. If the error satisfies a given tolerance, then return the approximation $I_T(h/2)$.

4. Else, decrease h and repeat on each sub-panel.

A simple (divide-and-conquer) version looks like:

```
function I = AdaptiveTrap(f,a,b,tol)
% Usage: I = AdaptiveTrap(f,a,b,tol)
% Purpose: Compute integral of f over (a,b) with absolute tolerance of tol
% Inputs: f -- function handle to f(x)
%         a -- lower limit
%         b -- upper limit
%         tol -- absolute tolerance
% Output: I = int_a^b f(x) dx to a tolerance of tol

% Call the basic trapezoidal rule with 1 and 2 panels
Ih  = TrapInt(f,a,b,1);
Ih2 = TrapInt(f,a,b,2);

if abs(Ih-Ih2)/3 < tol  % Check if error is satisfied
    I = Ih2;
else
    m = (b+a)/2;                            % Create two sub-panels and recurse
    Ileft  = AdaptiveTrap(f,a,m,tol/2); % Integrate left panel with half-tol
    Iright = AdaptiveTrap(f,m,b,tol/2); % Integrate right panel with half-tol
    I = Ileft + Iright;        % Return the sum of the left and right panels
end
end
```

The n-panel trapezoidal rule helper function is given as:

```
function I = TrapInt(f,a,b,n)
% Usage: I = TrapInt(f,a,b,n)
% Purpose: Integrate f(x) from a to b using an n-panel trapezoidal rule
% Inputs: f -- function handle to f(x)
%         a -- lower limit
%         b -- upper limit
% Output: I -- value of integral

x = linspace(a,b,n+1);

I = 0;

for i=1:n
    I = I + (x(i+1)-x(i)) * (f(x(i)) + f(x(i+1))) / 2;
end

end
```