ETH Zurich
Department of Mechanical and Process Engineering
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Nonlinear Continuum Mechanics
Exercise 9

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1 Useful Definitions or Concepts

1.1 Elastic constitutive laws

One general type of elastic material model is the one called Cauchy elastic material, which depend on only the current local deformation of the material and has no dependence on past history. This type is often to general and a slightly more restrictive model called Green elastic materials which are derivable from a scalar function (strain energy density function) are often used in mechanics.

- Cauchy elastic $\mathbf{S} = \hat{\mathbf{S}}(\text{deformation measure})$
- Green elastic $\mathbf{S} = \frac{\partial \Psi}{\partial \text{deformation measure})}$

The definition of Green elastic materials coincide with that of hyperelasticity. From hyperelasticity and thermodynamic consistency, one obtains,

$$\mathbf{P} = \frac{\partial \Psi}{\partial \mathbf{F}} \tag{1}$$

or

$$\mathbf{S} = 2\frac{\partial \Psi}{\partial \mathbf{C}} \,. \tag{2}$$

Objectivity requires that $\hat{\Psi}$ depend only the stretches or for example on the Right Cauchy deformation tensor,

$$\hat{\Psi}(\mathbf{C}) \,. \tag{3}$$

If the material is isotropic, then the Ψ can also be a function of the Left Cauchy deformation tensor,

$$\hat{\Psi}(\mathbf{b}) \tag{4}$$

or equivalently,

$$\hat{\Psi}(I_1(\mathbf{b}), I_2(\mathbf{b}), I_3(\mathbf{b})) \tag{5}$$

$$\hat{\Psi}(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C})) \,. \tag{6}$$

2 Applications

Problem:

Consider the Neo-Hookean model,

$$\Psi = c_1(I_1 - 3) + c_2(J^2 - 1) + c_3 \ln J \ . \tag{7}$$

What should c_1, c_2, c_3 be such that this model is consistent with linear elasticity? Express c_i $(i = 1, 2, 3)$ in terms of the Lame constants.

Solution:

The 2nd Piola-Kirchhoff stress tensor and Cauchy stress tensor can be computed by,

$$\mathbf{S} = 2c_1\mathbf{1} + 2c_2 2J\frac{1}{2}J\mathbf{C}^{-1} + 2c_3\frac{1}{J}\frac{1}{2}J\mathbf{C}^{-1} \tag{8}$$

$$\begin{aligned} \boldsymbol{\sigma} &= \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^T \\ &= \frac{2c_1}{J}\mathbf{b} + 2c_2 J\mathbf{1} + \frac{1}{J}c_3\mathbf{1} \ . \end{aligned} \tag{9}$$

Next consider the Cauchy stress tensor as a function of the displacement vector $\mathbf{u}$, and linearize at $\mathbf{0}$ displacement in the direction of $\mathbf{u}$. To compute this the linearization of $\frac{\mathbf{b}}{J}$ and J are computed. The definition of the Lin given a function $f(x)$ is, the linearization at y in the direction of u is,

$$\text{Lin} f(y)[u] = f(y) + Df(y)[u] \ . \tag{10}$$

Thus by defining,

$$\mathbf{F}(\mathbf{u}) = \mathbf{1} + \nabla_{\mathbf{X}}\mathbf{u} \tag{11}$$

$$\mathbf{b}(\mathbf{u}) = (\mathbf{1} + \nabla\mathbf{u})^T(\mathbf{1} + \nabla\mathbf{u}) \tag{12}$$

$$J(\mathbf{u}) = \det[\mathbf{1} + \nabla_{\mathbf{X}}\mathbf{u}] \tag{13}$$

$$\mathbf{F}(\eta) = \mathbf{1} + \eta\nabla_{\mathbf{X}}\mathbf{u} \tag{14}$$

$$\mathbf{b}(\eta) = (\mathbf{1} + \nabla\eta\mathbf{u})^T(\mathbf{1} + \nabla\eta\mathbf{u}) \tag{15}$$

$$J(\eta) = \det[\mathbf{1} + \nabla_{\mathbf{X}}\eta\mathbf{u}] \tag{16}$$

a slight abuse of notation, the linearization of the quantities involved are,

$$
\begin{aligned}
\mathrm{Lin}\frac{\mathbf{b}}{J}(\mathbf{0})[\mathbf{u}] &= \frac{\mathbf{b}}{J}(\mathbf{0}) + D\frac{\mathbf{b}}{J}(\mathbf{0})[\mathbf{u}] \\
&= \mathbf{1} + D\frac{\mathbf{b}}{J}(\mathbf{0})[\mathbf{u}] \\
D\frac{\mathbf{b}}{J}(\mathbf{0})[\mathbf{u}] &= \frac{d}{d\eta}\Big|_{\eta=0}\frac{\mathbf{b}(\eta)}{J(\eta)} \\
&= -\frac{1}{J^2(\eta)}\frac{dJ}{d\eta}\mathbf{b}(\eta) + \frac{1}{J(\eta)}\frac{d\mathbf{b}}{d\eta}\Big|_{\eta=0} \\
&= -\frac{1}{J^2}J(\eta)\mathbf{F}^{-T} : \frac{d\mathbf{F}}{d\eta}\mathbf{b} + \frac{1}{J}\left(\frac{d\mathbf{F}}{d\eta}\mathbf{F}^T + \mathbf{F}^T\frac{d\mathbf{F}}{d\eta}\right)\Big|_{\eta=0} \\
&= -(\mathbf{1} : \nabla_{\mathbf{X}}\mathbf{u})\mathbf{1} + \left(\nabla_{\mathbf{X}}\mathbf{u} + \nabla_{\mathbf{X}}\mathbf{u}^T\right) \\
&= -\mathrm{tr}\boldsymbol{\varepsilon}\mathbf{1} + 2\boldsymbol{\varepsilon} && (17) \\
\mathrm{lin}J(\mathbf{0})[\mathbf{u}] &= J(\mathbf{0}) + DJ(\mathbf{0})[\mathbf{u}] \\
&= 1 + \frac{d}{d\eta}\Big|_{\eta=0}J(\eta) \\
&= 1 + \mathrm{det}J(\eta)\mathbf{F}(\eta)^{-T} : \frac{d\mathbf{F}(\eta)}{d\eta}\Big|_{\eta=0} \\
&= 1 + \mathbf{1} : \nabla_{\mathbf{X}}\mathbf{u} \\
&= 1 + \mathrm{tr}\boldsymbol{\varepsilon} && (18) \\
\mathrm{lin}\frac{1}{J}(\mathbf{0})[\mathbf{u}] &= \frac{1}{J}(\mathbf{0}) + D\frac{1}{J}(\mathbf{0})[\mathbf{u}] \\
&= 1 - \frac{1}{J^2}\frac{d}{d\eta}J(\eta)\Big|_{\eta=0} \\
&= 1 - \mathrm{tr}\boldsymbol{\varepsilon} && (19)
\end{aligned}
$$

where,

$$
\begin{aligned}
\boldsymbol{\varepsilon} &= \frac{1}{2}\left(\mathrm{Div}\,[\mathbf{u}] + \mathrm{Div}\,[\mathbf{u}]^T\right) \\
&= \frac{1}{2}\left(\nabla_{\mathbf{X}}\mathbf{u} + \nabla_{\mathbf{X}}\mathbf{u}^T\right)\ . && (20)
\end{aligned}
$$

Using these relations, the linearization of $\boldsymbol{\sigma}$ with respect to $\mathbf{u}$ at $\mathbf{0}$ in the direction of $\mathbf{u}$ is,

$$
\mathrm{Lin}\boldsymbol{\sigma}(\mathbf{0})[\mathbf{u}] = 2c_1(\mathbf{1} - \mathrm{tr}\boldsymbol{\varepsilon}\mathbf{1} + 2\boldsymbol{\varepsilon}) + 2c_2(1 + \mathrm{tr}\boldsymbol{\varepsilon})\mathbf{1} + c_3(1 - \mathrm{tr}\boldsymbol{\varepsilon})\mathbf{1}\ . \tag{21}
$$

The stress-strain relationship for linear elastic materials is,

$$
\boldsymbol{\sigma}_{\text{linear}} = \lambda\mathrm{tr}[\boldsymbol{\varepsilon}]\mathbf{1} + 2\mu\boldsymbol{\varepsilon}\ . \tag{22}
$$

Comparing the two equations yields the relations,

$$
\begin{aligned}
2c_1 + 2c_2 + c_3 &= 0 && (23) \\
-2c_1 + 2c_2 - c_3 &= \lambda && (24) \\
4c_1 &= 2\mu\ . && (25)
\end{aligned}
$$

and thus,

$$c_1 = \frac{1}{2}\mu \tag{26}$$

$$c_2 = \frac{1}{4}\lambda \tag{27}$$

$$c_3 = -\mu - \frac{1}{2}\lambda\,. \tag{28}$$

Thus the constitutive equations has the form,

$$\Psi = \frac{1}{4}\lambda(J^2-1) - \left(\mu + \frac{1}{2}\lambda\right)\ln J + \frac{1}{2}\mu(I_1-3)\,. \tag{29}$$

Problem:

What is the response of a Neo-Hookean material to an in-plane shear motion,

$$\mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{30}$$

Solution:

$$\mathbf{b} = \begin{bmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{31}$$

The constitutive equation for the Neo-Hookean model obtained in the previous problem is,

$$\boldsymbol{\sigma} = \frac{\mu}{J}\mathbf{b} + \left(\frac{\lambda J}{2} - \frac{\mu}{J} - \frac{\lambda}{2J}\right)\mathbf{1}\,. \tag{32}$$

Inserting this deformation into the Neo-Hookean model yields,

$$\begin{aligned} \boldsymbol{\sigma} &= \mu\mathbf{b} - \mu\mathbf{1} \\ &= \begin{bmatrix} \mu\gamma^2 & \mu\gamma & 0 \\ \mu\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\,. \end{aligned} \tag{33}$$

A plot of the σ_{12} stress with respect to γ is shown in Figure 1.

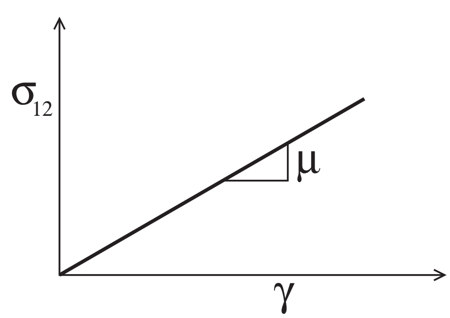

Figure 1: Shear stress versus deformation

Problem:

In an isotropic material, the principal directions of the Right Cauchy deformation tensor $\mathbf{C}$ and the 2nd Piola-Kirchhoff stress tensor coincide so that their spectral representations can be given as,

$$\mathbf{C} = \Sigma_{A=1}^{N} \lambda_A^2 \mathbf{N}_A \otimes \mathbf{N}_A \tag{34}$$

$$\mathbf{S} = \Sigma_{A=1}^{N} S_A \mathbf{N}_A \otimes \mathbf{N}_A \ . \tag{35}$$

If the material is hyperelastic with strain energy function Ψ the 2nd Piola-Kirchhoff stress tensor can be expressed as,

$$\mathbf{S} = 2\frac{\partial \Psi}{\partial \mathbf{C}} \ . \tag{36}$$

Given a strain energy function for an isotropic material with the arguments in terms of the principal values λ_i $(i = 1, 2, 3)$,

$$\Psi(\lambda_1, \lambda_2, \lambda_3) \tag{37}$$

express S_A in terms of Ψ and λ_A.

Solution:

The material time derivative of $\Psi, \mathbf{C}$ are,

$$\begin{aligned} \dot{\Psi} &= 2\frac{\partial \Psi}{\partial \mathbf{C}} : \frac{1}{2}\dot{\mathbf{C}} \\ &= \mathbf{S} : \dot{\mathbf{C}} \end{aligned} \tag{38}$$

$$\dot{\Psi} = \Sigma_A \frac{\partial \Psi}{\partial \lambda_A} \dot{\lambda}_A \tag{39}$$

$$\dot{\mathbf{C}} = \Sigma_A 2\lambda_A \dot{\lambda}_A \mathbf{N}_A \otimes \mathbf{N}_A + \lambda_A^2 \left[\dot{\mathbf{N}}_A \otimes \mathbf{N}_A + \mathbf{N}_A \otimes \dot{\mathbf{N}}_A \right] \ . \tag{40}$$

Define the orthogonal tensor $\mathbf{R}$ such that,

$$\mathbf{N}_A = \mathbf{R}\mathbf{E}_A \tag{41}$$

where $\mathbf{E}_A$ are unit vectors aligned with the Cartesian axis. Using this notation, the material time derivative of the principal directions becomes,

$$\begin{aligned} \dot{\mathbf{N}}_A &= \dot{\mathbf{R}}\mathbf{E}_A \\ &= \dot{\mathbf{R}}\mathbf{R}^T \mathbf{N}_A \\ &= \mathbf{\Omega}^{(L)} \mathbf{N}_A \end{aligned} \tag{42}$$

where,

$$\mathbf{\Omega}^{(L)} = \dot{\mathbf{R}}\mathbf{R}^T \tag{43}$$

is a skew tensor.(Verify by taking the material time derivative of $\mathbf{R}\mathbf{R}^T = \mathbf{1}$.) Since,

$$\begin{aligned} \mathbf{N}_A \cdot \dot{\mathbf{N}}_A &= \mathbf{N}_A \cdot \mathbf{\Omega}^{(L)} \mathbf{N}_A \\ &= 0 \end{aligned} \tag{44}$$

because $\mathbf{\Omega}^{(L)}$ is skew, the components of $\dot{\mathbf{C}}$ are,

$$\begin{aligned}\dot{C}_{AA} &= \mathbf{N}_A \cdot \dot{\mathbf{C}}\mathbf{N}_A \\ &= 2\lambda_A\dot{\lambda}_A \quad \text{No sum on A} \qquad (45)\\ \dot{C}_{AB} &= \mathbf{N}_A \cdot \dot{\mathbf{C}}\mathbf{N}_B \\ &= \lambda_B^2\Omega_{AB} + \lambda_A^2\Omega_{BA} \\ &= \Omega_{AB}(\lambda_B^2 - \lambda_A^2) \quad \text{A}\neq\text{B} \; . \qquad (46)\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{S} : \frac{1}{2}\dot{\mathbf{C}} &= (\Sigma S_A \mathbf{N}_A \otimes \mathbf{N}_A) : \frac{1}{2}\left(\Sigma_{B=1} 2\lambda_B\dot{\lambda}_B\mathbf{N}_B \otimes \mathbf{N}_B + \Sigma_{B\neq C}\Omega_{BC}(\lambda_C^2 - \lambda_B^2)\mathbf{N}_B \otimes \mathbf{N}_C\right) \\ &= \Sigma_A \lambda_A \dot{\lambda}_A \mathbf{N}_A \otimes \mathbf{N}_A \qquad (47)\end{aligned}$$

and equating the components with eqn. (39) yields,

$$\begin{aligned}\frac{\partial\Psi}{\partial\lambda_A}\dot{\lambda}_A &= \lambda_A\dot{\lambda}_A S_A \\ S_A &= \frac{1}{\lambda_A}\frac{\partial\Psi}{\partial\lambda_A} \; . \qquad (48)\end{aligned}$$

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3 Homework

3.1 Problem: Constitutive equations

Consider the Mooney-Rivlin material,

$$\Psi = c_1(I_1 - 3) + c_2(I_2 - 3) + c_3(J - 1)^2 \, . \tag{49}$$

Determine the conditions on c_1, c_2, and c_3 so that this model is consistent with isotropic linear elasticity.

3.2 Problem: Simple shear and stress

Make a similar plot to Figure 1 for the Mooney-Rivlin material in Problem #1 as well as for the Gent material,

$$\Psi = -\frac{\mu}{2} J_m \ln\left(1 - \frac{I_1 - 3}{J_m}\right) + \Gamma(J) \tag{50}$$

where J_m is a fixed material constant. Also, consider the Knowles material,

$$\Psi = \frac{\mu}{2b}\left[\left(1 + \frac{b}{n}(I_1 - 3)\right)^n - 1\right] + \Gamma(J) \tag{51}$$

where μ, b and n are material constants.

Comment on the effect of J_m in the Gent model and b and n in the Knowles model.

Remark: Knowledge of $\Gamma(\cdot)$ is not needed.

3.3 Problem: Principal values of stress

Knowing,

$$\mathbf{P} = \mathbf{F}\mathbf{S} \tag{52}$$

$$\boldsymbol{\tau} = \mathbf{F}\mathbf{S}\mathbf{F}^T \tag{53}$$

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{\tau} \tag{54}$$

find expressions for P_A, τ_A and σ_A, the principal values of $\mathbf{P}, \boldsymbol{\tau}$, and $\boldsymbol{\sigma}$.

Assume an Ogden model,

$$\Psi = \Sigma_{p=1}^{N} \frac{\mu_p}{\alpha_p}\left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p}\right) + \frac{1}{2}(J - 1)^2 \tag{55}$$

and compute an expression for σ_A. If one changes this to,

$$\Psi = \Sigma_{p=1}^{N} \frac{\mu_p}{\alpha_p}\left(\overline{\lambda}_1^{\alpha_p} + \overline{\lambda}_2^{\alpha_p} + \overline{\lambda}_3^{\alpha_p}\right) + \frac{1}{2}(J - 1)^2 \tag{56}$$

where $\overline{\lambda}_A = J^{-1/3}\lambda_A$, how does the expression for σ_A change?