ETH Zurich
Department of Mechanical and Process Engineering
Institute for Mechanical Systems
Center of Mechanics
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Nonlinear Continuum Mechanics
Exercise 8
Prof. Dr. Sanjay Govindjee

1 Useful Definitions or Concepts

1.1 Objectivity

Objectivity requires that given two observers O and O^+, and two events E_1 and E_2, they measure

- The same distance between events E_1 and E_2.
- The same orientation.
- The same time difference between the two events E_1 and E_2.
- The order in time which they are observed is the same.

This is the most general way that objectivity is defined and gives the flexibility that the two observers can be at two different times. In continuum mechanics, it is general enough to assume that the two observers measure events at the same time. In such a case the coordinates that the two observers measure $\mathbf{x}, \mathbf{x}^+$ are related by the equation,

$$\mathbf{x}^+ = \mathbf{Q}(t)\,(\mathbf{x} - \mathbf{o}) + \mathbf{c}(t) \tag{1}$$

where $\mathbf{o}$ defines some reference point, $\mathbf{Q}(t)$ an orthogonal tensor with positive determinant(to ensure that the two observers have the same orientation), $\mathbf{c}(t)$ a translation. For brevity in notation, the notation for the explicit time dependence in $\mathbf{Q}$ and $\mathbf{c}$ will be dropped.

1.1.1 Scalar values

For objectivity of a scalar value(such as the temperature of a body), the values that the two observers measure must be the same.

$$f^+ = f \tag{2}$$

1.1.2 Vectors

Vectors in a Euclidean space are the difference between points. Let us take two points $\mathbf{x}_1, \mathbf{x}_2$ and their corresponding points $\mathbf{x}_1^+, \mathbf{x}_2^+$. They are related by,

$$\mathbf{x}_1^+ = \mathbf{Q}(\mathbf{x}_1 - \mathbf{o}) + \mathbf{c} \tag{3}$$

$$\mathbf{x}_2^+ = \mathbf{Q}(\mathbf{x}_2 - \mathbf{o}) + \mathbf{c}\,. \tag{4}$$

The difference between these gives,

$$\mathbf{x}_2^+ - \mathbf{x}_1^+ = \mathbf{Q}(\mathbf{x}_2 - \mathbf{x}_1) \tag{5}$$

which implies that vectors between the two observers are related by,

$$\mathbf{a}^+ = \mathbf{Q}\mathbf{a}\,. \tag{6}$$

1.1.3 2nd Order tensors

A 2nd-order tensor can be represented as the sum of tensor products of vectors. By understanding how the tensor products are related between the observers enables one to understand the case for general 2nd-order tensors. Let $\mathbf{a}^+, \mathbf{b}^+$ be vectors observed by the O^+ observer and $\mathbf{a}, \mathbf{b}$ the corresponding vectors for observer O.

$$\begin{aligned} \mathbf{a}^+ \otimes \mathbf{b}^+ &= (\mathbf{Q}\mathbf{a}) \otimes (\mathbf{Q}\mathbf{b}) \\ &= \mathbf{Q}(\mathbf{a} \otimes \mathbf{b})\mathbf{Q}^T \end{aligned} \tag{7}$$

Thus in general, a 2nd-order tensor $\mathbf{d}^+$ is related to $\mathbf{d}$ by the relation,

$$\mathbf{d}^+ = \mathbf{Q}\mathbf{d}\mathbf{Q}^T . \tag{8}$$

1.2 Objectivity of quantaties in mechanics

In the continuum mechanics framework, the interest is to find the behaviour of the mapping φ that maps from a reference configuration $\mathcal{B}$ to $\mathcal{S}$. Given two observers O and O^+ even if they are observing the same deformation of the body, they will observe it in a different coordinate system and thus will have a seperate spatial configuration $\mathcal{S}$ and $\mathcal{S}^+$. Thus there exist two mapping $\varphi(\mathbf{X}, t)$ and $\varphi^+(\mathbf{X}, t)$. Since,

$$\mathbf{x}^+ = \mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{c} \tag{9}$$

the maps are related by,

$$\varphi^+(\mathbf{X}, t) = \mathbf{Q}(\varphi(\mathbf{X}, t) - \mathbf{o}) + \mathbf{c} . \tag{10}$$

It is assumed that at time $t = 0$,

$$\varphi(\mathbf{X}, 0) = \mathbf{X} \tag{11}$$

$$\varphi^+(\mathbf{X}, 0) = \mathbf{X} \tag{12}$$

which implies that,

$$\mathbf{Q}(0) = \mathbf{1} \tag{13}$$

$$\mathbf{c}(0) = \mathbf{0} . \tag{14}$$

In the spatial configuration, observers O and O^+ are observing the same event and phenomena and thus the statements made in the previous section must hold for quantaties measured by the two observers. Thus for spatial quantities,

1. Scalar $f^+ = f$
2. Vector $\mathbf{a}^+ = \mathbf{Q}\mathbf{a}$
3. 2nd-order Tensor $\mathbf{d}^+ = \mathbf{Q}\mathbf{d}\mathbf{Q}^T$

must hold for objectivity.

In the reference configuration, the observers O and O^+ measure the same quantity in the same coordinates $\mathbf{X}$ and thus the quantaties that they measure must agree. Denoting $F, \mathbf{A}, \mathbf{D}$ as a scalar, vector, and 2nd-Order tensor in the reference configuration, for reference quantaties,

1. Scalar $F^+ = F$

2. Vector $\mathbf{A}^+ = \mathbf{A}$
3. 2nd-order Tensor $\mathbf{D}^+ = \mathbf{D}$

must hold for objective quantaties.

For 2nd-order tensors a special situation can occure when it is defined in between the reference configuration and spatial configuration. The deformation gradient $\mathbf{F}$ and 1st Piola-Kirchhoff stress tensor is an example of such a case. Recall that these are called Two-point tensors. Let us observe what the transformation of such an element should be. Let $\mathbf{B}$ be a vector defined in the reference configuration and $\mathbf{a}, \mathbf{a}^+$ be a vector defined in the spatial configuration for observer O and O^+. Their tensor products are related by,

$$\begin{aligned} \mathbf{a}^+ \otimes \mathbf{B} &= (\mathbf{Q}\mathbf{a}) \otimes \mathbf{B} \\ &= \mathbf{Q}(\mathbf{a} \otimes \mathbf{B}) \ . \end{aligned} \tag{15}$$

Thus for a two-point tensor $\mathbf{F}$, objectivity requires,

$$\mathbf{F}^+ = \mathbf{Q}\mathbf{F} \ . \tag{16}$$

1.2.1 Examples

Let us look at quantaties in mechanics to see if they are objective.

- $\mathbf{F}$(deformation gradient):A two-point tensor

$$\begin{aligned} \mathbf{F}^+ &= \frac{\partial \boldsymbol{\varphi}^+(\mathbf{X}, t)}{\partial \mathbf{X}} \\ &= \frac{\partial \mathbf{x}^+(\mathbf{X}, t)}{\partial \mathbf{X}} \\ &= \frac{\partial \mathbf{x}^+}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \\ &= \mathbf{Q}\mathbf{F} \end{aligned} \tag{17}$$

  Thus it is objective.

- J(determinant of the deformation gradient): A scalar value

$$\begin{aligned} J^+ &= \det[\mathbf{Q}\mathbf{F}] \\ &= \det[\mathbf{Q}] \det[\mathbf{F}] \\ &= \det[\mathbf{F}] \\ &= J \end{aligned} \tag{18}$$

  Thus it is objective.

- $\mathbf{C}$(Right Cauchy deformation tensor): Defined in reference configuration.

$$\begin{aligned} \mathbf{C}^+ &= \mathbf{F}^{+,T}\mathbf{F}^+ \\ &= (\mathbf{Q}\mathbf{F})^T(\mathbf{Q}\mathbf{F}) \\ &= \mathbf{F}^T\mathbf{Q}^T\mathbf{Q}\mathbf{F} \\ &= \mathbf{F}^T\mathbf{F} \\ &= \mathbf{C} \end{aligned} \tag{19}$$

  Thus it is objective.

- $\mathbf{b}$(Left Cauchy deformation tensor): Defined in spatial configuration.

$$\begin{aligned} \mathbf{b}^+ &= \mathbf{F}^+\mathbf{F}^{+,T} \\ &= (\mathbf{QF})(\mathbf{QF})^T \\ &= \mathbf{QFF}^T\mathbf{Q}^T \\ &= \mathbf{QFF}^T\mathbf{Q} \\ &= \mathbf{QbQ}^T \end{aligned} \tag{20}$$

  Thus it is objective.

- $\mathbf{l}$(The spatial velocity gradient): Defined in the spatial configuration.

$$\begin{aligned} \mathbf{l}^+ &= \dot{\mathbf{F}^+}\mathbf{F}^{+,-1} \\ &= \dot{\mathbf{QF}}(\mathbf{QF})^{-1} \\ &= \left(\dot{\mathbf{Q}}\mathbf{F} + \mathbf{Q}\dot{\mathbf{F}}\right)\mathbf{F}^{-1}\mathbf{Q}^T \\ &= \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{QlQ}^T \\ &= \mathbf{Q}\left(\mathbf{Q}^T\dot{\mathbf{Q}} + \mathbf{l}\right)\mathbf{Q}^T \end{aligned} \tag{21}$$

  Thus it is not objective.

- $\boldsymbol{\sigma}$(Cauchy stress tensor): Defined in spatial configuration. Let $d\mathbf{f}$ be a force acting on a surface element in the spatial configuration. $d\mathbf{f}^+$ and $d\mathbf{f}$ are related by,

$$\begin{aligned} d\mathbf{f}^+ &= \mathbf{Q}d\mathbf{f} \\ \mathbf{t}^+\, da^+ &= \mathbf{Qt}\, da \\ \boldsymbol{\sigma}^{+,T}\mathbf{n}^+\, da^+ &= \mathbf{Q}\boldsymbol{\sigma}^T\mathbf{n}\, da \\ \boldsymbol{\sigma}^{+,T}J^+\mathbf{F}^{+,-T}\mathbf{N}\, dA &= \mathbf{Q}\boldsymbol{\sigma}^T J\mathbf{F}^{-T}\mathbf{N}\, dA \\ \boldsymbol{\sigma}^{+,T}J(\mathbf{QF})^{-T}\mathbf{N}\, dA &= \mathbf{Q}\boldsymbol{\sigma}^T J\mathbf{F}^{-T}\mathbf{N}\, dA \\ \boldsymbol{\sigma}^{+,T}J\mathbf{QF}^{-T}\mathbf{N}\, dA &= \mathbf{Q}\boldsymbol{\sigma}^T J\mathbf{F}^{-T}\mathbf{N}\, dA \\ \boldsymbol{\sigma}^{+,T}\mathbf{Q} &= \mathbf{Q}\boldsymbol{\sigma}^T \\ \boldsymbol{\sigma}^{+,T} &= \mathbf{Q}\boldsymbol{\sigma}^T\mathbf{Q}^T \end{aligned} \tag{22}$$

  Thus it is objective.

- $\mathbf{P}$(1st Piola-Kirchhoff stress tensor): Two point tensor.

$$\begin{aligned} \mathbf{P}^+ &= J^+\boldsymbol{\sigma}^+\mathbf{F}^{+,-T} \\ &= J\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T(\mathbf{QF})^{-T} \\ &= J\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T\mathbf{QF}^{-T} \\ &= \mathbf{Q}(J\boldsymbol{\sigma}\mathbf{F}^{-T}) \\ &= \mathbf{QP} \end{aligned} \tag{23}$$

  Thus it is objective.

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2 Applications of definitions or concepts

Problem:

Let the reference body $\mathcal{B}$ be a square with unit sides. Assume that the material is homogeneous(does not depend on $\mathbf{X}$) and isotropic (the material response is the same in all directions). For such a material, by application of the representation theorem, the constitutive law can be written as,

$$\boldsymbol{\sigma} = \alpha_0 \mathbf{1} + \alpha_1 \mathbf{b} + \alpha_2 \mathbf{b}^2 \tag{24}$$

where,

$$\alpha_i = \alpha_i(\mathbf{I_b}, \mathbf{II_b}, \mathbf{III_b}) \tag{25}$$

$$\mathbf{I_b} = \mathrm{tr}\,[\mathbf{b}] \tag{26}$$

$$\mathbf{II_b} = \frac{1}{2}\left((\mathrm{tr}\,[\mathbf{b}])^2 - \mathrm{tr}\left[\mathbf{b}^2\right]\right) \tag{27}$$

$$\mathbf{III_b} = \det[\mathbf{b}] \tag{28}$$

A simple shear motion will be applied,

$$\mathbf{x} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{X}\,. \tag{29}$$

Compute the stresses, and confirm the Poynting effect.

Solution:

First let us compute $\mathbf{F},\ \mathbf{b},\ \mathbf{b}^2, \boldsymbol{\sigma}$.

$$\mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{30}$$

$$\mathbf{b} = \begin{bmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{31}$$

$$\mathbf{b}^2 = \begin{bmatrix} (1+\gamma^2)^2+\gamma^2 & \gamma(1+\gamma^2)+\gamma & 0 \\ \gamma(1+\gamma^2)+\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{32}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \alpha_0+\alpha_1(1+\gamma^2)+\alpha_2(\gamma^4+3\gamma^2+1) & \alpha_1\gamma+\alpha_2(\gamma^3+2\gamma) & 0 \\ \alpha_1\gamma+\alpha_2(\gamma^3+2\gamma) & \alpha_0+\alpha_1+\alpha_2(1+\gamma^2) & 0 \\ 0 & 0 & \alpha_0+\alpha_1+\alpha_2 \end{bmatrix} \tag{33}$$

If $\boldsymbol{\sigma}$ is normalized so that $\boldsymbol{\sigma}(\mathbf{F}=\mathbf{1})=\mathbf{0}$, then

$$\alpha_0+\alpha_1+\alpha_2 = 0\,. \tag{34}$$

With this expression,

$$\boldsymbol{\sigma} = \begin{bmatrix} \alpha_1(\gamma^2)+\alpha_2(\gamma^4+3\gamma^2) & \alpha_1\gamma+\alpha_2(\gamma^3+2\gamma) & 0 \\ \alpha_1\gamma+\alpha_2(\gamma^3+2\gamma) & \alpha_2(\gamma^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}\,. \tag{35}$$

It can be seen that simple shear in the finite deformation case does yields $\sigma_{11} \neq 0$, $\sigma_{22} \neq 0$. This is called the Poynting effect. Additionally, irrelevant of the material, as long as it is isotropic,

$$\sigma_{11} - \sigma_{22} \quad = \quad \gamma\sigma_{12} \,. \tag{36}$$

Problem:

Given a cylinder with radius R, length L, and density ρ_0, let us spin this around the axis at a constant rate of ω. Eventually the inertial forces will balance with the internal forces and the system will reach a steady state. The cylinder is assumed homogenous isotropic incompressible. In this case the motion of the body can be expressed as,

$$x_1 = \lambda^{-\frac{1}{2}} \left(X_1 \cos(\omega t) - X_2 \sin(\omega t) \right) \tag{37}$$

$$x_2 = \lambda^{-\frac{1}{2}} \left(X_1 \sin(\omega t) + X_2 \cos(\omega t) \right) \tag{38}$$

$$x_3 = \lambda X_3 \,. \tag{39}$$

The steady state assumption allows us to assume that λ will not depend on time,

$$\dot{\lambda} = 0 \,. \tag{40}$$

The constitutive equation is given as,

$$\boldsymbol{\sigma} = -p\mathbf{1} + \mu\mathbf{b} \tag{41}$$

which is a slight modification of the representation theorem. p denotes the pressure and is included in the constitutive equation because there is a incompressibility constraint on the material. Recall that if one had a cube of material that is incompressible, no matter how much pressure is applied, the deformation would not change. This implies that a pressure-deformation relationship cannot be obtained in the case of an incompressible material, and the pressure must be defined by the boundary conditions.

The boundary conditions applied on the cylinder is zero traction on the curved side of the cylinder, and the resultant force on the top and bottom of the cylinder is zero,

$$\mathbf{f}_{\text{top}} = \int_{\text{top surface}} \mathbf{t} \, da = 0 \tag{42}$$

$$\mathbf{f}_{\text{bot}} = \int_{\text{bottom surface}} \mathbf{t} \, da = 0 \,. \tag{43}$$

Determine λ, where a $\lambda > 1$ will imply that the cylinder will have stretched in the axial direction, and $\lambda < 1$ will imply that the cylinder is compressed in the axial direction.

Solution:

Let us first compute the deformation gradient, stress, and spatial acceleration.

$$\mathbf{F} = \begin{bmatrix} \lambda^{-\frac{1}{2}}\cos(\omega t) & -\lambda^{-\frac{1}{2}}\sin(\omega t) & 0 \\ \lambda^{-\frac{1}{2}}\sin(\omega t) & \lambda^{-\frac{1}{2}}\cos(\omega t) & 0 \\ 0 & 0 & \lambda X_3 \end{bmatrix} \tag{44}$$

$$\begin{aligned} \mathbf{b} &= \mathbf{F}^T\mathbf{F} \\ &= \begin{bmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix} \end{aligned} \tag{45}$$

$$\begin{aligned} \boldsymbol{\sigma} &= -p\mathbf{1} + \mu\mathbf{b} \\ &= \begin{bmatrix} -p+\mu\lambda^{-1} & 0 & 0 \\ 0 & -p+\mu\lambda^{-1} & 0 \\ 0 & 0 & -p+\mu\lambda^2 \end{bmatrix} \end{aligned} \tag{46}$$

$$\mathbf{V} = \begin{bmatrix} \omega\lambda^{-\frac{1}{2}}\left(X_1\sin(\omega t) - X_2\cos(\omega t)\right) \\ \omega\lambda^{-\frac{1}{2}}\left(X_1\cos(\omega t) - X_2\sin(\omega t)\right) \\ 0 \end{bmatrix} \tag{47}$$

$$\mathbf{A} = \begin{bmatrix} -\omega^2\lambda^{-\frac{1}{2}}\left(X_1\cos(\omega t) - X_2\sin(\omega t)\right) \\ -\omega^2\lambda^{-\frac{1}{2}}\left(X_1\sin(\omega t) + X_2\cos(\omega t)\right) \\ 0 \end{bmatrix} \tag{48}$$

$$\mathbf{a} = \begin{bmatrix} -\omega^2 x_1 \\ -\omega^2 x_2 \\ 0 \end{bmatrix} \tag{49}$$

Next the equilibrium equation is used,

$$\operatorname{div}[\boldsymbol{\sigma}] = \rho\mathbf{a}\ . \tag{50}$$

Inserting the expressions above under the assumption of homogenous deformations,

$$\frac{\partial\lambda}{\partial x_i} = 0 \tag{51}$$

yields,

$$\frac{\partial p}{\partial x_1} = \rho\omega^2 x_1 \tag{52}$$

$$\frac{\partial p}{\partial x_2} = \rho\omega^2 x_2 \tag{53}$$

$$\frac{\partial p}{\partial x_3} = 0\ . \tag{54}$$

The third equation implies that there is no x_3 dependence on p and thus from the first equation,

$$p(x_1, x_2) = \frac{1}{2}\rho\omega^2 x_1^2 + \phi_1(x_2, t)\ . \tag{55}$$

Inserting this into the second equation yields,

$$\frac{\partial\phi_1}{\partial x_2} = \rho\omega^2 x_2 \tag{56}$$

and integration of this equation yields,

$$\phi_1 = \frac{1}{2}\rho\omega^2 x_2^2 + \phi(t) \; . \tag{57}$$

Thus,

$$\begin{aligned} p(x_1, x_2) &= \frac{1}{2}\rho\omega^2(x_1^2 + x_2^2) + \phi(t) \\ &= \frac{1}{2}\rho\omega^2 r^2 + \phi(t) \end{aligned} \tag{58}$$

where $r^2 = x_1^2 + x_2^2$. The function $\phi(t)$ is determined by the boundary conditions on the curved sides of the cylinder($r = \lambda^{-\frac{1}{2}}R$).

$$\mathbf{t}_{\text{side}} = \boldsymbol{\sigma}_{\text{side}} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \tag{59}$$

$$||\mathbf{t}_{\text{side}}|| = 0 \tag{60}$$

This yields,

$$-p + \mu\lambda^{-1} = 0 \tag{61}$$

and thus,

$$\phi(t) = \frac{1}{2}\rho\omega^2 R^2\lambda^{-1} + \mu\lambda^{-1} \tag{62}$$

which determines the pressure as a function of λ. Finally to determine λ, the boundary condition at the top and bottom of the cylinder are utilized.

$$\begin{aligned} \mathbf{t}_{\text{top}} &= \boldsymbol{\sigma}\mathbf{e}_3 \\ &= (-p(r) + \mu\lambda^2)\mathbf{e}_3 \\ \mathbf{f}_{\text{top}} &= \int_{\text{top}} -p(r) + \mu\lambda^2 \; da\mathbf{e}_3 \\ &= \int_0^{R\lambda^{-\frac{1}{2}}} -p(r) + \mu\lambda^2 \; 2\pi r dr\mathbf{e}_3 \\ &= \frac{R^2\mu\pi}{\lambda^2}\left[\lambda^3 - \left(1 - \frac{1}{4}\rho\omega^2\frac{R^2}{\mu}\right)\right]\mathbf{e}_3 && (63) \\ &= \mathbf{0} \; . && (64) \end{aligned}$$

Since $\lambda > 0$,

$$\lambda = \left[1 - \frac{1}{4}\rho\omega^2\frac{R^2}{\mu}\right]^{1/3} < 1 \; . \tag{65}$$

Thus the cylinder will contract in the axial direction and get larger in the radial direction.

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3 Homework

3.1 Problem:Objectivity

Show that the quantaties $\mathbf{R}, \mathbf{U}, \mathbf{V}$(the tensors that arise from the polar decompositions) are objective.

We have seen that the rate of deformation tensor $\mathbf{d}$, which is a spatial tensor, is an objective rate. Equivalently we can see that the material time derivative of the Green-Lagrange tensor is an objective rate since,

$$\mathbf{E}^{+} = \mathbf{E} \tag{66}$$

and thus,

$$\dot{\mathbf{E}^{+}} = \dot{\mathbf{E}}\,. \tag{67}$$

Similarly we can define objective stress rates. Given the Cauchy stress spatial tensor $\boldsymbol{\sigma}$,

$$\boldsymbol{\sigma}^{+} = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{T} \tag{68}$$

where $\mathbf{Q}$ is the rotation tensor presented in the change of observer equation. Show that the material time derivative of $\boldsymbol{\sigma}$ is not objective, i.e. show that,

$$\dot{\overline{\boldsymbol{\sigma}^{+}}} \neq \mathbf{Q}\dot{\boldsymbol{\sigma}}\mathbf{Q}^{T}\,. \tag{69}$$

Then show that,

$$\text{Jaumann}[\boldsymbol{\sigma}] = \dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w} \tag{70}$$
$$\text{Oldroyd}[\boldsymbol{\sigma}] = \dot{\boldsymbol{\sigma}} - \mathbf{l}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{l}^{T} \tag{71}$$

which are the Jaumann stress rate and Oldroyd stress rate of the Cauchy stress tensor, are objective stress rates.

Using these objective strain and stress rates, one can describe constitutive equations which must be denoted in their rate form.

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3.2 Problem: A rubber balloon

Consider a spherical balloon, initially with radius R and thickness T. Assume that $T << R$. The balloon is inflated to a current radius of r and current thickness of t. The pressure of the gas inside of the balloon is p and the pressure outside is zero.

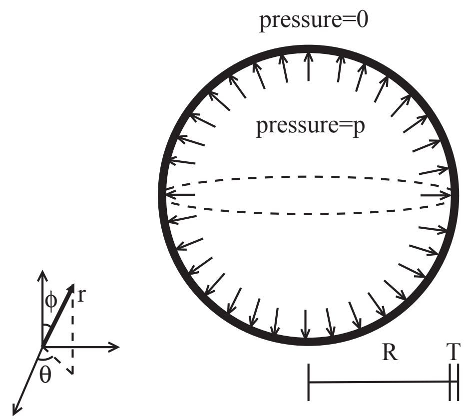

Figure 1: Schematic of balloon

Part 1

Compute the principal stretches in polar coordinates ($\lambda_r,\ \lambda_\phi,\ \lambda_\theta$) using symmetry arguments.

Hint:

Recall that the physical meaning of the principal stretch is,

$$\lambda = \frac{\text{Current length}}{\text{Initial length}} . \tag{72}$$

Since we have spherical symmetry $\lambda_\phi = \lambda_\theta$.

Part 2

Obtain the relation,

$$\sigma = \frac{pr}{2t} \tag{73}$$

under the assumption of thin shell $t << r$. Here, σ is the membrane tension and p is the internal pressure.

Hint:

Cut a sphere in half and compute equilibrium between the membrane forces and pressure over the section.

Part 3

The material of the balloon will be modeled as incompressible with constitutive equation,

$$\boldsymbol{\sigma} = -q\mathbf{1} + \mu\mathbf{b} \,, \tag{74}$$

where q is the pressure distribution inside of the thin layer of the balloon material, and μ is a material constant. Similar to the example for the rotating cylinder, recall that when there is an incompressibility constraint, the pressure contribution to the stress tensor $\boldsymbol{\sigma}$ cannot be determined by a constitutive equation and must be solved from the boundary conditions of the problem. The q denoted here is this quantity and differs from the pressure p of the gas inside of the balloon.

Express p in terms of T, μ, R, and $\frac{V}{V_0}$ where V is the current volume of the balloon and V_0 is the reference volume of the balloon. Plot p as a function of $\frac{V}{V_0}$ and comment on the meaning of the curve you obtain. You should see a peak in the middle.

Hint:

Since we are considering spherical coordinates, the components of $\boldsymbol{\sigma}$ are,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi\phi} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix} \tag{75}$$

and the components of $\mathbf{b}$ are defined similarly. There is no rotational deformation in this problem, thus $\mathbf{R}$ in the polar decomposition of $\mathbf{F}$ is equal to identity. Additionally the pricipal directions of the deformation do not change,

$$\mathbf{U} = \begin{bmatrix} \lambda_r & 0 & 0 \\ 0 & \lambda_\theta & 0 \\ 0 & 0 & \lambda_\phi \end{bmatrix} . \tag{76}$$

Using the assumption of incompressibility($\det \mathbf{F} = 1$), express $\boldsymbol{\sigma}$ in terms of q and λ_θ. Then using the assumption that t is small, which implies that $\sigma_{rr} = 0$, express $\boldsymbol{\sigma}$ in terms of just λ_θ. Since the membrane stress σ is equal to $\sigma_{\theta\theta}$, obtain an expression of p in terms of T, R, μ, and λ_θ. Finally express p in terms of T, μ, R, and $\frac{V}{V_0}$.