ETH Zurich
Department of Mechanical and Process Engineering
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Nonlinear Continuum Mechanics
Exercise 7

Institute for Mechanical Systems
Center of Mechanics

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1 Useful Definitions or Concepts

1.1 Stress tensors

The stress tensors that have been introduced are,

- $\boldsymbol{\sigma}$: Cauchy stress tensor. Obtained from Cauchy's theorem as a tensor which gives us the stress state in the deformed body. It is defined in the spatial configuration. The traction vector obtained from the application of the surface normal in the spatial(deformed) configuration $\mathbf{n}$,

$$\mathbf{t} = \boldsymbol{\sigma}^T \mathbf{n} \tag{1}$$

is called the Cauchy traction vector. It represents,

$$\mathbf{t} = \frac{\text{Force acting on the surface element in spatial configuration}}{\text{Area in the deformed configuration}} \tag{2}$$

Since it gives us the actual stress of the body, it is called the true stress in engineering.

- $\boldsymbol{\tau}$: Kirchhoff stress tensor. It is defined as,

$$\boldsymbol{\tau} = J\boldsymbol{\sigma} \, . \tag{3}$$

It is convenient to define this quantatity since it is related to the later defined 2nd Piola-Kirchhoff stress tensor $\mathbf{S}$ through a push-forward.

$$\varphi_*(\mathbf{S}) = \boldsymbol{\tau} \, . \tag{4}$$

- $\mathbf{P}$: 1st Piola-Kirchhoff stress tensor. It is defined as,

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T} \, . \tag{5}$$

This definition is motivated by the following argument. Let $d\mathbf{f}$ be a force vector acting on an infinitesimal surface area da in the spatial (deformed) configuration. By definition of the Cauchy traction vector which is the force acting on unit deformed area,

$$\mathbf{t} = \frac{d\mathbf{f}}{da} \, . \tag{6}$$

This implies,

$$\begin{aligned} d\mathbf{f} &= \mathbf{t} \, da \\ &= \boldsymbol{\sigma}^T \mathbf{n} \, da \\ &= \boldsymbol{\sigma}^T J\mathbf{F}^{-T}\mathbf{N} \, dA \\ &= \mathbf{P}\mathbf{N} \, dA \\ &= \mathbf{T} \, dA \, . \end{aligned}$$

Thus $\mathbf{P}$ is defined so that a vector $\mathbf{T}$ which is called the 1st Piola-Kirchhoff stress vector can be defined in terms of the tensor $\mathbf{P}$ and the surface normal $\mathbf{N}$ in the reference (undeformed) configuration. Manipulating the obtained relationship, one obtains,

$$\mathbf{T} = \frac{d\mathbf{f}}{dA} \tag{7}$$

and thus the 1st Piola-Kirchhoff stress vector represents,

$$\mathbf{T} \quad = \quad \frac{\text{Force acting on the surface element in spatial configuration}}{\text{Area in the undeformed configuration}} \tag{8}$$

This stress is called the nominal stress in engineering and thus the corresponding tensor $\mathbf{P}$ is also called the nominal stress tensor. This tensor is convenient since it supplies us with the direction of the force by using the surface normals $\mathbf{N}$ in the undeformed configuration, of which we know the direction. There is no way of knowing what the surface normals of the deformed configuration $\mathbf{n}$ are until we have solved the problem!

The tensor $\mathbf{P}$ is special in the sense that it takes vectors defined in the reference(undeformed) configuration to vectors defined in the spatial(deformed) configuration. For this reason it is called a two-point tensor. The deformation gradient $\mathbf{F}$, which maps vectors from the reference configuration to the spatial configuration is also an example of a two-point tensor.

Using this stress tensor, the equilibrium equation,

$$\text{div}\left[\boldsymbol{\sigma}^T\right] + \rho\mathbf{b} \quad = \quad \rho\mathbf{a} \tag{9}$$

can be written in the terms of quantaties defined in the reference configuration,

$$\text{Div}\left[\mathbf{P}\right] + \rho_0\mathbf{B} \quad = \quad \rho_0\mathbf{A} \tag{10}$$

where $J\rho = \rho_0$ and $\mathbf{B}(\mathbf{X}) = \mathbf{b}(\boldsymbol{\varphi}(\mathbf{x}))$.

- $\mathbf{S}$: 2nd Piola-Kirchhoff stress tensor. It is defined as,

$$\mathbf{S} \quad = \quad J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T} \tag{11}$$

$$\quad = \quad \mathbf{F}^{-1}\mathbf{P}\,. \tag{12}$$

This tensor is defined for the reasons that it is a material tensor which is defined in the reference configuration and that it is the work conjugate of the Green-Lagrange strain tensor $\mathbf{E}$ enabling one to define diffent constitutive equations. It also has the atractive property that it is symmetric (if $\boldsymbol{\sigma}$ is symmetric).

Analogously to the 1st Piola-Kirchhoff stress vector, we can define a 2nd Piola-Kirchhoff stress vector,

$$\mathbf{T}_S \quad = \quad \mathbf{S}\mathbf{N} \tag{13}$$

but it has little physical interpretation and thus not used. The expression yields,

$$\begin{aligned} \mathbf{T}_s \quad &= \quad \mathbf{F}^{-1}\mathbf{P}\mathbf{N} \\ &= \quad \mathbf{F}^{-1}\mathbf{T} \\ &= \quad \mathbf{F}^{-1}\frac{d\mathbf{f}}{dA} \\ &= \quad \frac{\mathbf{F}^{-1}d\mathbf{f}}{dA} \end{aligned} \tag{14}$$

which is,

$$\mathbf{T}_S \quad = \quad \frac{\text{Force acting on the surface element in spatial configuration pulled back to the reference configuration}}{\text{Area in the undeformed configuration}} \tag{15}$$

1.2 Theorem of expended power

$$\frac{D}{Dt}\int_{P_t}\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\,d\mathbf{x}+\int_{P_t}\boldsymbol{\sigma}:\mathbf{d}\,d\mathbf{x} \quad = \quad \int_{P_t}\mathbf{b}\cdot\mathbf{v}\,d\mathbf{x}+\int_{\partial P_t}\mathbf{t}\cdot\mathbf{v}\,da \tag{16}$$

This theorem tells us that,

$$\begin{aligned}&\text{The change in kinetic energy } + \text{ Internal mechanical work}\\ &\quad = \text{ Work done by the body forces } + \text{ Work done by the surface forces}\end{aligned} \tag{17}$$

This theorem can also be expressed in terms of spatial quantaties. See the Applications section.

1.3 The mechanical boundary value problem

Let a body $\mathcal{B}$ be given with body forces $\rho_0\mathbf{B}$ in $\mathcal{B}$, surface tractions $\bar{\mathbf{T}}$ on $\partial_{\mathbf{T}}\mathcal{B}\subset\partial\mathcal{B}$, and prescribed deformation $\bar{\boldsymbol{\varphi}}$ on $\partial_{\boldsymbol{\varphi}}\mathcal{B}$. Find $\boldsymbol{\varphi}(\mathbf{X},t)$ satisfying the equations,

$$\begin{array}{ll}\text{Div}\,[\mathbf{P}(\boldsymbol{\varphi})]+\rho_0\mathbf{B}=\rho_0\ddot{\boldsymbol{\varphi}} & \text{in } \mathcal{B} \text{ and } \forall t\\ \boldsymbol{\varphi}=\bar{\boldsymbol{\varphi}} & \text{on } \partial_{\boldsymbol{\varphi}}\mathcal{B} \text{ and } \forall t\\ \mathbf{P}(\boldsymbol{\varphi})\mathbf{N}=\bar{\mathbf{T}} & \text{on } \partial_{\mathbf{T}}\mathcal{B} \text{ and } \forall t\\ \boldsymbol{\varphi}(\mathbf{X},0)=\boldsymbol{\varphi}_0(\mathbf{X}),\ \dot{\boldsymbol{\varphi}}(\mathbf{X},0)=\mathbf{V}_0(\mathbf{X}) & \text{in } \mathcal{B}\end{array} \tag{18}$$

where $\boldsymbol{\varphi}_0$ is the prescribed deformation, $\bar{\mathbf{T}}$ is the prescribed traction, $\boldsymbol{\varphi}_0$ is the initial deformation and $\mathbf{V}_0$ is the initial velocity field.

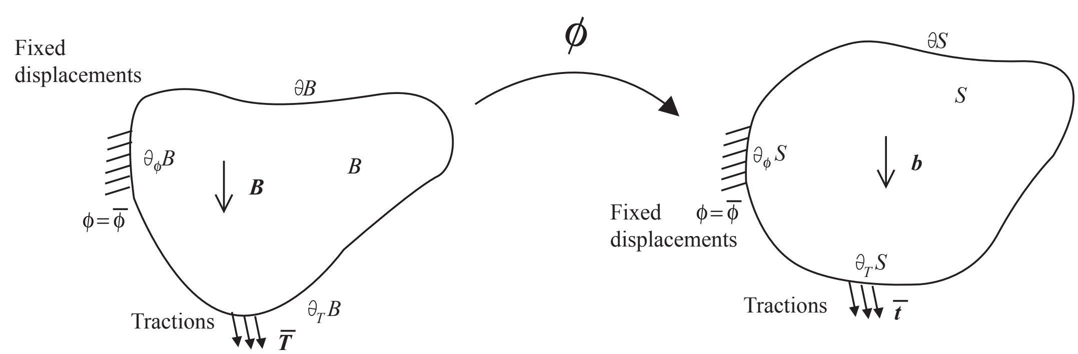

Figure 1: Mechanical problem

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2 Applications

Problem:

Prove Piola's identity.

$$\text{Div}\left[J\mathbf{F}^{-T}\right] = 0 \tag{19}$$
$$\text{div}\left[\frac{1}{J}\mathbf{F}^{T}\right] = 0\,. \tag{20}$$

Solution:

The brute force method is conducted as follows.

$$\begin{aligned}\text{Div}\left[J\mathbf{F}^{-T}\right] &= \left(J\left(\mathbf{F}^{-T}\right)_{iA}\right)_{,A} \\ &= J_{,A}\mathbf{F}^{-1}_{Ai} + J\mathbf{F}^{-1}_{Ai,A}\,. \end{aligned} \tag{21}$$

The gradient of the Jacobian is evaluated as,

$$\begin{aligned} J_{,A} &= \frac{\partial J}{\partial X_A} \\ &= \frac{\partial J}{\partial F_{jB}}\frac{\partial F_{jB}}{\partial X_A} \\ &= J\mathbf{F}^{-T}_{jB}F_{jB,A}\,. \end{aligned} \tag{22}$$

The divergence of the inverse transpose of the deformation gradient is obtained by the following steps.

$$\begin{aligned} \delta_{ij} &= \mathbf{F}_{iA}\mathbf{F}^{-1}_{Aj} \\ \delta_{ij,B} &= \left(\mathbf{F}_{iA}\mathbf{F}^{-1}_{Aj}\right)_{,B} \\ 0 &= \mathbf{F}_{iA,B}\mathbf{F}^{-1}_{Aj} + \mathbf{F}_{iA}\mathbf{F}^{-1}_{Aj,B} \\ \mathbf{F}_{iA,B}\mathbf{F}^{-1}_{Aj} &= -\mathbf{F}_{iA}\mathbf{F}^{-1}_{Aj,B} \\ \mathbf{F}_{iA}\mathbf{F}^{-1}_{Aj,B} &= -\mathbf{F}_{iA,B}\mathbf{F}^{-1}_{Aj} \\ \mathbf{F}^{-1}_{Bi}\mathbf{F}_{iA}\mathbf{F}^{-1}_{Aj,B} &= -\mathbf{F}^{-1}_{Bi}\mathbf{F}_{iA,B}\mathbf{F}^{-1}_{Aj} \quad \text{(Multiply both sides by } \mathbf{F}\text{)} \\ \delta AB\mathbf{F}^{-1}_{Aj,B} &= -\mathbf{F}^{-1}_{Bi}\mathbf{F}_{iA,B}\mathbf{F}^{-1}_{Aj} \\ \mathbf{F}^{-1}_{Aj,A} &= -\mathbf{F}^{-1}_{Bi}\mathbf{F}_{iA,B}\mathbf{F}^{-1}_{Aj} \end{aligned} \tag{23}$$

Assuming continuous second derivatives for the motion,

$$\begin{aligned} \mathbf{F}_{iA,B} &= \frac{\partial^2 \boldsymbol{\varphi}_i}{\partial X_B X_A} \\ &= \frac{\partial^2 \boldsymbol{\varphi}_i}{\partial X_A X_B} \\ &= \mathbf{F}_{iB,A} \end{aligned} \tag{24}$$

insertion of the two relations in eqn. (21) yields,

$$\begin{aligned} \text{Div}\left[J\mathbf{F}^{T}\right] &= J\mathbf{F}^{-T}_{jB}F_{jB,A}\mathbf{F}^{-1}_{Ai} - J\mathbf{F}^{-1}_{Bi}\mathbf{F}_{iA,B}\mathbf{F}^{-1}_{Aj} \\ &= J\mathbf{F}^{-T}_{jB}F_{jA,B}\mathbf{F}^{-1}_{Ai} - J\mathbf{F}^{-1}_{Bi}\mathbf{F}_{iA,B}\mathbf{F}^{-1}_{Aj} \\ &= 0 \end{aligned} \tag{25}$$

The proof for the case where div $[\cdot]$ is the following.

$$
\begin{aligned}
\operatorname{div}\left[\frac{1}{J}\mathbf{F}^T\right] &\Leftrightarrow \left(\frac{1}{J}F_{iA}\right)_{,i} \\
&= \left(\frac{1}{J}\right)_{,i} F_{iA} + \frac{1}{J}F_{iA,i} \\
&= -\frac{1}{J^2}J_{,i}F_{iA} + \frac{1}{J}F_{iA,i} \\
&= -\frac{1}{J^2}\frac{\partial J}{\partial F_{jB}}\frac{\partial F_{jB}}{\partial x_i}F_{iA} + \frac{1}{J}F_{iA,i} \\
&= -\frac{1}{J^2}J\mathbf{F}^{-1}_{Bj}F_{jB,i}F_{iA} + \frac{1}{J}F_{iA,i} \\
&= -\frac{1}{J}\frac{\partial X_B}{\partial x_j}\frac{\partial^2 \boldsymbol{\varphi}_j}{\partial x_i X_B}\frac{\partial x_i}{\partial X_A} + \frac{1}{J}F_{iA,i} \\
&= -\frac{1}{J}\frac{\partial^2 \boldsymbol{\varphi}_j}{\partial x_j X_A} + \frac{1}{J}F_{iA,i} \\
&= -\frac{1}{J}F_{jA,j} + \frac{1}{J}F_{iA,i} \\
&= 0
\end{aligned}
\tag{26}
$$

A more elegant method to obtain these relations is presented here.

$$
\begin{aligned}
0 &= \int_{P_t} \nabla \cdot 1 \, d\mathbf{x} \\
&= \int_{\partial P_t} \mathbf{n} \, da \\
&= \int_{P_0} J\mathbf{F}^{-T}\mathbf{N} \, da \\
&= \int_{P_0} \operatorname{Div}\left[J\mathbf{F}^{-T}\right] \, d\mathbf{x}
\end{aligned}
\tag{27}
$$

Since the domain of integration $P_0$is arbitrary, the integrand must equal zero by the localization theorem. The case for div $\left[\frac{1}{J}\mathbf{F}^T\right]$ can be proven similarly.

Problem:

Let a motion be given for a 2D-beam,

$$\mathbf{x} = \mathbf{A}\mathbf{X} \tag{28}$$

$$\mathbf{A} = \mathbf{R}\mathbf{U} \tag{29}$$

$$\mathbf{F} = \mathbf{A} \tag{30}$$

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{31}$$

$$\mathbf{U} = \begin{bmatrix} \frac{l}{L} & 0 \\ 0 & \frac{h}{H} \end{bmatrix}. \tag{32}$$

The beam initally has width H and length L, and in the deformed state has width h and length l. Assume that in the deformed configuration, there is a force of f applied to the beam in the direction of the axis. The force is assumed to be distributed evenly on the surface. Calculate the Cauchy stress tensor and First and Second Piola Kirchhoff tensors.

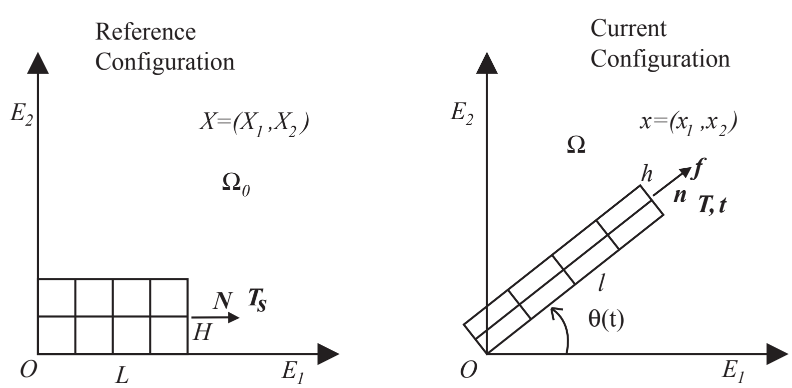

Figure 2: Mechanical problem

Solution:

It is clear that the Cauchy stress tensor in the coordinates aligned with axis of the beam is given as,

$$\boldsymbol{\sigma}' = \begin{bmatrix} \frac{f}{h} & 0 \\ 0 & 0 \end{bmatrix}. \tag{33}$$

In the global coordinates, by an application of a coordinate transformation, the tensor has the form,

$$\begin{aligned} \boldsymbol{\sigma} &= \mathbf{R}\boldsymbol{\sigma}'\mathbf{R}^T \\ &= \frac{f}{h}\begin{bmatrix} \cos^2\theta & \sin\cos\theta \\ \sin\cos\theta & \sin^2\theta \end{bmatrix}. \end{aligned} \tag{34}$$

Since,

$$\mathbf{F}^{-1} = \mathbf{U}^{-1}\mathbf{R}^T \tag{35}$$

$$\mathbf{F}^{-T} = \mathbf{R}\mathbf{U}^{-1} \tag{36}$$

$$J = \frac{lh}{LH} \tag{37}$$

The first and second Piola-Kirchhoff tensors are,

$$\begin{aligned}\mathbf{P} &= J\boldsymbol{\sigma}\mathbf{F}^{-T} \\ &= J\mathbf{R}\boldsymbol{\sigma}'\mathbf{R}^T\mathbf{R}\mathbf{U}^{-1} \\ &= J\mathbf{R}\boldsymbol{\sigma}'\mathbf{U}^{-1} \\ &= \begin{bmatrix} \frac{f}{H}\cos\theta & 0 \\ \frac{f}{H}\sin\theta & 0 \end{bmatrix}\end{aligned} \tag{38}$$

$$\begin{aligned}\mathbf{S} &= \mathbf{F}^{-1}\mathbf{P} \\ &= \left(\mathbf{U}^{-1}\mathbf{R}^T\right)\left(J\mathbf{R}\boldsymbol{\sigma}'\mathbf{U}^{-1}\right) \\ &= J\mathbf{U}^{-1}\boldsymbol{\sigma}'\mathbf{U}^{-1} \\ &= \begin{bmatrix} \frac{Lf}{lH} & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned} \tag{39}$$

Define a normal to the end of the beam in both the reference and spatial configurations.

$$\mathbf{N} = [1, 0]^T \tag{40}$$

$$\mathbf{n} = [\cos\theta, \sin\theta]^T \tag{41}$$

The tractions can be computed as follows.

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \frac{f}{h}\mathbf{n} \tag{42}$$

$$\mathbf{T} = \mathbf{P}\mathbf{N} = \frac{f}{H}\mathbf{n} \tag{43}$$

$$\mathbf{T}_S = \mathbf{S}\mathbf{N} = \frac{Lf}{lH}\mathbf{N} \tag{44}$$

Since the force applied on the surface can be represented as $f\mathbf{n}$,

$$\mathbf{t}h = f\mathbf{n} \tag{45}$$

$$\mathbf{T}H = f\mathbf{n}\,. \tag{46}$$

$\mathbf{t}$ is the force per unit deformed area which points in the direction of the actual force. Thus the Cauchy stress tensor maps normal vectors in the spatial configuration to traction vectors defined in the spatial configuration.

$\mathbf{T}$ is the force per unit undeformed area which points in the direction of the actual force. Thus the 1st Piola-Kirchhoff stress tensor maps normal vectors in the reference configuration to traction vectors defined in the spatial configuration. Tensors of this nature which map vectors from one configuration to another are called two point tensors. The deformation gradient is another example of a two point tensor.

$\mathbf{T}_S$ is the force per unit undeformed area which points in the direction of the pull back of the traction vector $\mathbf{t}$. Thus intuitively this quantity is difficult to interpret physically.

Problem:

Show that the two forms of equilibrium,

$$\begin{aligned}\text{div}\left[\boldsymbol{\sigma}^T\right] + \rho\mathbf{b} &= \rho\mathbf{a} \qquad (47)\\ \text{Div}\left[\mathbf{P}\right] + \rho_0\mathbf{B} &= \rho_0\mathbf{A} \qquad (48)\end{aligned}$$

where $\mathbf{B}(\mathbf{X},t) = \mathbf{b}(\boldsymbol{\varphi}(\mathbf{X},t),t)$ and $\mathbf{A}(\mathbf{X},t) = \mathbf{a}(\boldsymbol{\varphi}(\mathbf{X},t),t)$ are equivalent.

Solution:

One can prove this through either the partial differential equation form or the integral form. First the partial differential case is given. From the Piola identity div $\left[\frac{1}{J}\mathbf{F}^T\right]$,

$$\begin{aligned}\text{div}\left[\boldsymbol{\sigma}^T\right] &\Leftrightarrow (\sigma_{ji})_{,j}\\ &= \left(\frac{1}{J}P_{iA}F_{jA}\right)_{,j}\\ &= (P_{iA})_{,j}\frac{1}{J}F_{jA} + \left(\frac{1}{J}F_{jA}\right)_{,j}P_{iA}\\ &= (P_{iA})_{,j}\frac{1}{J}F_{jA} + \left(\text{div}\left[\frac{1}{J}F^T\right]\right)_A P_{iA}\\ &= (P_{iA})_{,j}\frac{1}{J}F_{jA}\\ &= \frac{1}{J}\frac{\partial P_{iA}}{\partial x_j}\frac{\partial x_j}{\partial X_A}\\ &= \frac{1}{J}\frac{\partial P_{iA}}{\partial X_A}\\ &\Leftrightarrow \frac{1}{J}\text{Div}\left[\mathbf{P}\right]\ . \qquad (49)\end{aligned}$$

Inserting this into eqn. (47),

$$\begin{aligned}\frac{1}{J}\text{Div}\left[\mathbf{P}\right] + \rho\mathbf{b} &= \rho\mathbf{a}\\ \text{Div}\left[\mathbf{P}\right] + J\rho\mathbf{b} &= J\rho\mathbf{a}\\ \text{Div}\left[\mathbf{P}\right] + \rho_0\mathbf{B} &= \rho_0\mathbf{A}\ . \qquad (50)\end{aligned}$$

Next the integral form is given. Let P_t be an arbitrary domain.

$$
\begin{aligned}
\int_{P_t} \rho \mathbf{a}\, d\mathbf{x} &= \int_{P_t} \operatorname{div}\left[\boldsymbol{\sigma}^T\right] + \rho \mathbf{b}\, d\mathbf{x} \\
&= \int_{\partial P_t} \boldsymbol{\sigma}^T \mathbf{n}\, da + \int_{P_t} \rho \mathbf{b}\, d\mathbf{x} \\
&= \int_{\partial P_0} \boldsymbol{\sigma}^T \mathbf{F}^{-T} \mathbf{N}\, J\, dA + \int_{P_t} \rho \mathbf{b}\, d\mathbf{x} \\
&= \int_{\partial P_0} \mathbf{P}\mathbf{N}\, dA + \int_{P_t} \rho \mathbf{b}\, d\mathbf{x} \\
\int_{P_0} \rho \mathbf{A} J\, d\mathbf{X} &= \int_{P_0} \operatorname{Div}\left[\mathbf{P}\right]\, d\mathbf{X} + \int_{P_0} \rho \mathbf{B} J\, d\mathbf{X} \\
\int_{P_0} \rho_0 \mathbf{A}\, d\mathbf{X} &= \int_{P_0} \operatorname{Div}\left[\mathbf{P}\right]\, d\mathbf{X} + \int_{P_0} \rho_0 \mathbf{B}\, d\mathbf{X} \qquad (51)
\end{aligned}
$$

Since this holds for any P_0, the integrand must be zero and thus the desired relation is obtained.

Problem:

Show that the equation,

$$\frac{D}{Dt}\int_{P_t}\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\,d\mathbf{x}+\int_{P_t}\boldsymbol{\sigma}:\mathbf{d}\,d\mathbf{x} \;=\; \int_{P_t}\mathbf{b}\cdot\mathbf{v}\,d\mathbf{x}+\int_{\partial P_t}\mathbf{t}\cdot\mathbf{v}\,da \tag{52}$$

holds true. Also show that,

$$\int_{P_t}\boldsymbol{\sigma}:\mathbf{d}\,d\mathbf{x} = \int_{P_t}\boldsymbol{\sigma}:\mathbf{l}\,d\mathbf{x} \tag{53}$$

$$= \int_{P_0}\boldsymbol{\tau}:\mathbf{d}\,d\mathbf{X} \tag{54}$$

$$= \int_{P_0}\mathbf{P}:\dot{\mathbf{F}}\,d\mathbf{X} \tag{55}$$

$$= \int_{P_0}\mathbf{S}:\frac{1}{2}\dot{\mathbf{C}}\,d\mathbf{X} \tag{56}$$

$$= \int_{P_0}\mathbf{S}:\dot{\mathbf{E}}\,d\mathbf{X}\,. \tag{57}$$

Solution:

Take the innerproduct of the equilibrium equation and velocity field and integrate over an arbitrary domain P_t.

$$\begin{aligned}
0 &= \int_{P_t}\left[\operatorname{div}\left[\boldsymbol{\sigma}^T\right]+\rho(\mathbf{b}-\mathbf{a})\right]\cdot\mathbf{v}\,d\mathbf{x}\\
&= \int_{P_t}\operatorname{div}\left[\boldsymbol{\sigma}^T\right]\cdot\mathbf{v}+\rho(\mathbf{b}-\mathbf{a})\cdot\mathbf{v}\,d\mathbf{x}\\
&= \int_{P_t}\sigma_{ji,j}v_i+\rho(\mathbf{b}-\mathbf{a})\cdot\mathbf{v}\,d\mathbf{x}\\
&= \int_{P_t}(\sigma_{ji}v_i)_j-\sigma_{ji}v_{i,j}+\rho(\mathbf{b}-\mathbf{a})\cdot\mathbf{v}\,d\mathbf{x}\\
\int_{P_t}\rho(\mathbf{b}-\mathbf{a})\cdot\mathbf{v}\,d\mathbf{x} &= \int_{\partial P_t}\sigma_{ji}v_in_j\,da-\int_{P_t}\sigma_{ji}v_{i,j}\,d\mathbf{x}\\
&= \int_{\partial P_t}t_iv_i\,da-\int_{P_t}\sigma_{ji}l_{ij}\,d\mathbf{x}\\
&= \int_{\partial P_t}\mathbf{t}\cdot\mathbf{v}\,da-\int_{P_t}\boldsymbol{\sigma}:\mathbf{l}\,d\mathbf{x}\\
&= \int_{\partial P_t}\mathbf{t}\cdot\mathbf{v}\,da-\int_{P_t}\boldsymbol{\sigma}:\mathbf{d}\,d\mathbf{x}
\end{aligned} \tag{58}$$

Here the symmetry of $\boldsymbol{\sigma}$ has been utilised. Since,

$$\begin{aligned}
\frac{D}{Dt}\int_{P_t}\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\,d\mathbf{x} &= \frac{D}{Dt}\int_{P_0}\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}J\,d\mathbf{X} \\
&= \frac{D}{Dt}\int_{P_0}\frac{1}{2}\rho_0\mathbf{v}\cdot\mathbf{v}\,d\mathbf{X} \\
&= \int_{P_0}\frac{1}{2}\rho_0 2\dot{\mathbf{v}}\cdot\mathbf{v}\,d\mathbf{X} \\
&= \int_{P_0}\rho_0\mathbf{a}\cdot\mathbf{v}\,d\mathbf{X} \\
&= \int_{P_0}\rho\mathbf{a}\cdot\mathbf{v}J\,d\mathbf{X} \\
&= \int_{P_t}\rho\mathbf{a}\cdot\mathbf{v}\,d\mathbf{x}
\end{aligned} \tag{59}$$

combination with eqn. (58) gives the desired result.

The stress power can be manipulated to give different expressions.

$$\begin{aligned}
\int_{P_t}\boldsymbol{\sigma}:\mathbf{d}\,d\mathbf{x} &= \int_{P_0}\boldsymbol{\sigma}:\mathbf{d}J\,d\mathbf{X} \\
&= \int_{P_0}\boldsymbol{\tau}:\mathbf{d}\,d\mathbf{X} \\
&= \int_{P_t}\boldsymbol{\sigma}:\mathbf{l}\,d\mathbf{x} \quad \text{(Cauchy stress is symmetric)} \\
&= \int_{P_0}\boldsymbol{\sigma}:\dot{\mathbf{F}}\mathbf{F}^{-1}J\,d\mathbf{X} \\
&= \int_{P_0}J\boldsymbol{\sigma}\mathbf{F}^{-T}:\dot{\mathbf{F}}\,d\mathbf{X} \\
&= \int_{P_0}\mathbf{P}:\dot{\mathbf{F}}\,d\mathbf{X} \\
&= \int_{P_0}\mathbf{F}\mathbf{S}:\dot{\mathbf{F}}\,d\mathbf{X} \\
&= \int_{P_0}\mathbf{S}:\mathbf{F}^T\dot{\mathbf{F}}\,d\mathbf{X} \\
&= \int_{P_0}\mathbf{S}:\left(\mathbf{F}^T\dot{\mathbf{F}}\right)_{\text{symm}}\,d\mathbf{X} \quad \text{(Symmetry of S)} \\
&= \int_{P_0}\mathbf{S}:\frac{1}{2}\dot{\mathbf{C}}\,d\mathbf{X} \\
&= \int_{P_0}\mathbf{S}:\dot{\mathbf{E}}
\end{aligned} \tag{60}$$

From this one observes that $(\boldsymbol{\sigma},\mathbf{d})$, $(\mathbf{P},\mathbf{F})$, and $(\mathbf{S},\mathbf{E})$ are conjugate pairs.

Solution:

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3 Homework

3.1 Principle of virtual work

The mechanical problem in the reference configuration is given as follows.

Let a a body $\mathcal{B}$ (reference configuration) be given with, loading from body forces $\mathbf{B}$ (force per unit mass) defined in $\mathcal{B}$ and surface tractions $\mathbf{T}$ defined on $\partial_T\mathcal{B} \subset \partial\mathcal{B}$, and fixed deformations $\bar{\boldsymbol{\varphi}}$ on $\partial_{\boldsymbol{\varphi}}\mathcal{B}$. Find a mapping $\boldsymbol{\varphi}$ that satisfies,

$$\begin{array}{lll} \text{Equilibrium} & \mathrm{Div}\,[\mathbf{P}(\boldsymbol{\varphi})] + \rho_0\mathbf{B} = \rho_0\ddot{\boldsymbol{\varphi}} & \text{in } \mathcal{B} \text{ and } \forall t \\ \text{Displacement boundary condition} & \boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}} & \text{on } \partial_{\boldsymbol{\varphi}}\mathcal{B} \text{ and } \forall t \\ \text{Force boundary condition} & \mathbf{P}(\boldsymbol{\varphi})\mathbf{N} = \bar{\mathbf{T}} & \text{on } \partial_{\mathbf{T}}\mathcal{B} \text{ and } \forall t \\ \text{Initial condition} & \boldsymbol{\varphi}(\mathbf{X},0) = \boldsymbol{\varphi}_0(\mathbf{X}),\ \dot{\boldsymbol{\varphi}}(\mathbf{X},0) = \mathbf{V}_0(\mathbf{X}) & \text{in } \mathcal{B} \end{array} \tag{61}$$

Here $\mathbf{P}$ is assumed to depend on the mapping $\boldsymbol{\varphi}$, so in short the deformation. For the static problem it can be simplified

to the following statement.

Mechanical problem: Let a body $\mathcal{B}$ (reference configuration) be given with loading from body forces $\mathbf{B}$ (force per unit mass) defined in $\mathcal{B}$ and surface tractions $\mathbf{T}$ defined on $\partial_T\mathcal{B} \subset \partial\mathcal{B}$, and fixed deformations $\bar{\boldsymbol{\varphi}}$ on $\partial_{\boldsymbol{\varphi}}\mathcal{B}$. Find a mapping $\boldsymbol{\varphi}$ that satisfies,

$$\begin{array}{lll} \text{Equilibrium} & \mathrm{Div}\,[\mathbf{P}(\boldsymbol{\varphi})] + \rho_0\mathbf{B} = 0 & \text{in } \mathcal{B} \\ \text{Displacement boundary condition} & \boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}} & \text{on } \partial_{\boldsymbol{\varphi}}\mathcal{B} \\ \text{Force boundary condition} & \mathbf{P}(\boldsymbol{\varphi})\mathbf{N} = \bar{\mathbf{T}} & \text{on } \partial_{\mathbf{T}}\mathcal{B} \end{array} \tag{62}$$

This problem can also be stated in another form, which is also called the principle of virtual work.

Mechanical problem (Principle of virtual work): Let a body $\mathcal{B}$ (reference configuration) be given with loading from body forces $\mathbf{B}$ (force per unit mass) defined in $\mathcal{B}$ and surface tractions $\mathbf{T}$ defined on $\partial_T\mathcal{B} \subset \partial\mathcal{B}$, and fixed deformations $\bar{\boldsymbol{\varphi}}$ on $\partial_{\boldsymbol{\varphi}}\mathcal{B}$. Find a mapping $\boldsymbol{\varphi}$ that satisfies,

$$\int_B \mathbf{P} : \mathrm{Grad}[\delta\boldsymbol{\varphi}]\, d\mathbf{X} \quad = \quad \int_B \rho_0\mathbf{B}\cdot\delta\boldsymbol{\varphi}\, d\mathbf{X} + \int_{\partial_{\mathbf{T}}B} \bar{\mathbf{T}}\cdot\delta\boldsymbol{\varphi}\, dA \quad (\forall\delta\boldsymbol{\varphi} \text{ such that } \delta\boldsymbol{\varphi}(\mathbf{X}) = 0 \text{ on } \partial_{\boldsymbol{\varphi}}\mathbf{B}) \tag{63}$$

In numerical methods such as the finite element method, it is this equation which is descretized and solved. In this problem the steps to obtain this form will be outlined.

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Part 1.

Given eqn. (62.1), take the inner product of this equation with a function $\delta\boldsymbol{\varphi}$, integrate over the body $\mathcal{B}$ to show that the relation,

$$\int_B \mathbf{P} : \text{Grad}[\delta\boldsymbol{\varphi}]\, d\mathbf{X} \quad = \quad \int_B \rho_0 \mathbf{B}\cdot\delta\boldsymbol{\varphi}\, d\mathbf{X} + \int_{\partial B} \mathbf{T}\cdot\delta\boldsymbol{\varphi}\, dA \tag{64}$$

holds.

(Hint: Follow a procedure shown similarly in the lecture.)

Part 2.

Using the displacement and force boundary conditions of eqn. (62), starting from eqn. (64), argue that eqn. (63) holds.

Part 3.

Recall the definition of the directional derivative. Given a scalar-valued vector argument function $f(\mathbf{v})$, the directional derivative at $\mathbf{x}$ in the direction of $\mathbf{h}$ was defined as,

$$Df(\mathbf{x})[\mathbf{h}] := \frac{d}{d\eta}\Big|_{\eta=0} f(\mathbf{x}+\eta\mathbf{h})\ . \tag{65}$$

We can conduct a similar operation on the deformation gradient $\mathbf{F}$,

$$\mathbf{F}(\boldsymbol{\varphi}) := \frac{\partial\boldsymbol{\varphi}}{\partial\mathbf{X}} = \text{Grad}[\boldsymbol{\varphi}] \tag{66}$$

which depends on the deformation mapping $\boldsymbol{\varphi}$. The directional derivative of $\mathbf{F}(\boldsymbol{\varphi})$ in the direction of $\delta\boldsymbol{\varphi}$ at $\boldsymbol{\varphi}$ is defined as

$$\begin{aligned} \delta\mathbf{F} &:= D\mathbf{F}(\boldsymbol{\varphi})[\delta\boldsymbol{\varphi}] \\ &= \frac{d}{d\eta}\Big|_{\eta=0}\mathbf{F}(\boldsymbol{\varphi}+\eta\delta\boldsymbol{\varphi}) \\ &= \frac{d}{d\eta}\Big|_{\eta=0}\text{Grad}[\boldsymbol{\varphi}+\eta\delta\boldsymbol{\varphi}] \\ &= \frac{d}{d\eta}\Big|_{\eta=0}\text{Grad}[\boldsymbol{\varphi}]+\eta\text{Grad}[\delta\boldsymbol{\varphi}] \\ &= \text{Grad}[\delta\boldsymbol{\varphi}]\ . \end{aligned} \tag{67}$$

Show that,

$$\delta\mathbf{C} \quad = \quad 2\text{sym}[\mathbf{F}^T\text{Grad}[\delta\boldsymbol{\varphi}]] \tag{68}$$

$$\delta\mathbf{E} \quad = \quad \text{sym}[\mathbf{F}^T\text{Grad}[\delta\boldsymbol{\varphi}]]. \tag{69}$$

Here sym implies that the symmetric part is extracted.

$$\text{sym}[\mathbf{A}] \quad = \quad \frac{1}{2}(\mathbf{A}+\mathbf{A}^T)$$

(Hint: $\mathbf{C}(\boldsymbol{\varphi}+\eta\delta\boldsymbol{\varphi}) = \mathbf{F}^T(\boldsymbol{\varphi}+\eta\delta\boldsymbol{\varphi})\mathbf{F}(\boldsymbol{\varphi}+\eta\delta\boldsymbol{\varphi})$)

Part 4.

Show that eqn. (64) implies,

$$\int_B \mathbf{S} : \delta\mathbf{E}\, d\mathbf{X} \quad = \quad \int_B \rho_0 \mathbf{B} \cdot \delta\boldsymbol{\varphi}\, d\mathbf{X} + \int_{\partial_{\mathbf{T}} B} \bar{\mathbf{T}} \cdot \delta\boldsymbol{\varphi}\, dA \tag{70}$$

Part 5.

Show that eqn. (64) implies,

$$\int_S \boldsymbol{\sigma} : \text{sym}[\text{grad}[\delta\boldsymbol{\varphi}]]\, d\mathbf{x} \quad = \quad \int_S \rho\mathbf{b} \cdot \delta\boldsymbol{\varphi}\, d\mathbf{x} + \int_{\partial_{\mathbf{T}} S} \bar{\mathbf{t}} \cdot \delta\boldsymbol{\varphi}\, da \tag{71}$$

where $\mathcal{S} = \boldsymbol{\varphi}(\mathcal{B})$ is the mapped body in the spatial configuration, $J\rho = \rho_0$, $\mathbf{b}(\mathbf{x}) = \mathbf{B}(\boldsymbol{\varphi}^{-1}(\mathbf{x}))$, and $\bar{\mathbf{t}}da = \bar{\mathbf{T}}dA$.

(Hint: From chain rule,

$$\begin{aligned} \text{grad}[\delta\boldsymbol{\varphi}] &= \frac{\partial \delta\boldsymbol{\varphi}}{\partial \mathbf{x}} \\ &= \frac{\partial \delta\boldsymbol{\varphi}}{\partial \mathbf{X}} \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \\ &= \text{Grad}[\delta\boldsymbol{\varphi}]\mathbf{F}^{-1} \end{aligned}$$

)

3.2 Thermodynamic inequalities

In class, the following inequality was obtained from the Clausius-Duhem inequality.

$$-\rho\dot{\Psi} - \rho\eta\dot{T} + \boldsymbol{\sigma} : \mathbf{d} - \frac{1}{T}\text{grad}[T] \cdot \mathbf{q} \geq 0 \tag{72}$$

Here Ψ is Helmholtz's free energy and η is the entropy per unit mass. This relationship is given in terms of spatial quantaties. First show that,

$$\text{Grad}[T] \quad = \quad \mathbf{F}^T \text{grad}[T] \tag{73}$$

is true. Then using the definition of the heat flux in the material configuration $\mathbf{Q}$ as,

$$\mathbf{Q} \quad = \quad J\mathbf{F}^{-1}\mathbf{q} \tag{74}$$

show the material representation of the inequality,

$$-\rho_0\dot{\Psi} - \rho_0\eta\dot{T} + \mathbf{P} : \dot{\mathbf{F}} - \frac{1}{T}\text{Grad}[T] \cdot \mathbf{Q} \geq 0$$
$$-\rho_0\dot{\Psi} - \rho_0\eta\dot{T} + \mathbf{S} : \dot{\mathbf{E}} - \frac{1}{T}\text{Grad}[T] \cdot \mathbf{Q} \geq 0 \ .$$

(Hint: Multiply eqn. (72) by J and manipulate).