ETH Zurich | Institute for Mechanical Systems
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Nonlinear Continuum Mechanics
Exercise 6 | Prof. Dr. Sanjay Govindjee

1 Useful Definitions or Concepts

1.1 Change of variable in the integration

Let a body $\mathcal{B}$ undergo a motion $\boldsymbol{\varphi}$. Let the body in the spatial configuration be denoted as $\mathcal{S}$. The integration of a quantity $\mathbf{f}(\mathbf{x})$ in the spatial configuration can be transformed to an integration over the reference configuration through the change of variable,

$$\int_S \mathbf{f}(\mathbf{x})\, d\mathbf{x} \quad = \quad \int_B \mathbf{f}(\boldsymbol{\varphi}(\mathbf{X}))J\, d\mathbf{X} \tag{1}$$

where $J = \det\mathbf{F}$ and $\mathbf{F} = \text{Grad}\boldsymbol{\varphi}$.

1.2 Conservation of mass or Continuity equation

This equation is derived from the idea that in a body $\mathcal{B}$ that does not have mass generation or reduction, the mass is a conserved quantaty. A mapping $\boldsymbol{\varphi}$ from the referene configuration $\mathcal{B}$ to the spatial configuration $\mathcal{S}$ is assumed. Let $P_t \subset \mathcal{S}$ be mapped from $P_0 \subset \mathcal{B}$.

$$\int_{P_0} \rho_0\, d\mathbf{X} \quad = \quad \int_{P_t} \rho\, d\mathbf{x} \tag{2}$$

Here ρ_0 is the density defined in the reference configuration and ρ is the density defined in the spatial configuration. Transformation of the spatial integral to a material integral and application of the localization theorem yields,

$$\rho_0 \quad = \quad \rho J \tag{3}$$

where $J = \det F$. ρ_0 is independent of time and a material time derivative of the expression above gives in the material description,

$$\dot{\rho} + \rho \text{div}\mathbf{v} \quad = \quad 0 \tag{4}$$

and in the spatial description,

$$\rho' + \text{div}(\rho\mathbf{v}) \quad = \quad 0\,. \tag{5}$$

1.3 Balance of linear momentum and angular momentum

The balance of linear momentum states that the change in linear momentum is equal to the applied external force. Given the same assumption as above,

$$\frac{D}{Dt}\int_{P_t} \rho\mathbf{v}\, d\mathbf{x} \quad = \quad \int_{P_t} \rho\mathbf{b}\, d\mathbf{x} + \int_{\partial P_t} \mathbf{t}\, da \tag{6}$$

where $\mathbf{b}$ is the body force per unit mass and $\mathbf{t}$ are the surface tractions. Taking the material time derivative of the right hand side gives,

$$\begin{aligned} \frac{D}{Dt}\int_{P_0} \rho J \mathbf{v}\, d\mathbf{X} &= \int_{P_t} \rho \mathbf{b}\, d\mathbf{x} + \int_{\partial P_t} \mathbf{t}\, da \\ \frac{D}{Dt}\int_{P_0} \rho_0 \mathbf{v}\, d\mathbf{X} &= \\ \int_{P_0} \rho_0 \dot{\mathbf{v}}\, d\mathbf{X} &= \\ \int_{P_0} \rho J \dot{\mathbf{v}}\, d\mathbf{X} &= \\ \int_{P_t} \rho \dot{\mathbf{v}}\, d\mathbf{x} &= \quad . \end{aligned} \tag{7}$$

The balance of angular momentum states that the change in angular momentum is equal to the applied external moment. Given the same assumption as above,

$$\frac{D}{Dt}\int_{P_t} \mathbf{x} \times \rho \mathbf{v}\, d\mathbf{x} = \int_{P_t} \mathbf{x} \times \rho \mathbf{b}\, d\mathbf{x} + \int_{\partial P_t} \mathbf{x} \times \mathbf{t}\, da \tag{8}$$

where $\mathbf{b}$ is the body force per unit mass and $\mathbf{t}$ are the surface tractions. Taking the material time derivative of the right hand side gives,

$$\int_{P_t} \mathbf{x} \times \rho \dot{\mathbf{v}}\, d\mathbf{x} = \int_{P_t} \mathbf{x} \times \rho \mathbf{b}\, d\mathbf{x} + \int_{\partial P_t} \mathbf{x} \times \mathbf{t}\, da\,. \tag{9}$$

1.4 Cauchy's theorem

Given a continuous traction field $\mathbf{t}(\mathbf{x}, \mathbf{n})$ and body force field (per unit mass) $\mathbf{b}(\mathbf{x})$,

1. Global linear momentum balance,
2. Global angular momentum balance,

holds if and only if there exists a tensor $\boldsymbol{\sigma}(\mathbf{x})$ such that,

1. $\mathbf{t}(\mathbf{x}, \mathbf{n}) = \boldsymbol{\sigma}^T(\mathbf{x})\mathbf{n}$,
2. $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$,
3. $\text{div}(\boldsymbol{\sigma}^T) + \rho\mathbf{b} = \rho\mathbf{a}$.

Since the Cauchy stress tensor $\boldsymbol{\sigma}$ is symmetric, it allows a spectral decomposition,

$$\boldsymbol{\sigma} = \Sigma_{A=1}^{3} \lambda_A \hat{\mathbf{n}}_A \otimes \hat{\mathbf{n}}_A\,. \tag{10}$$

The values λ_A are the maximal stress values and the corresponding directions $\hat{\mathbf{n}}_A$ are the corresponding directions. The $\hat{\mathbf{n}}_A$ are orthonormal and it can clearly be seen that there is no shear stress in this basis. Compared to the case of the Right Cauchy-Green tensor, it is not positive definite. It if were positive definite, there would be no zero or negative(compressive) stress state which is absurd.

2 Applications of definitions or concepts

Problem:

Show Newton's first law for a body from the continuum mechanics standpoint. Given a body $\mathcal{B}$, a motion $\mathcal{S}$, the body in the spatial configuration $\mathcal{S}$, and its center of mass $\overline{\mathbf{x}}$,

$$\overline{\mathbf{x}} = \frac{1}{m(S)} \int_S \rho \mathbf{x} \, d\mathbf{x} \,, \tag{11}$$

the governing motion for the body can be expressed in terms of the center of mass as,

$$m(S)\overline{\mathbf{x}} = \mathbf{f} \,. \tag{12}$$

Here $\mathbf{f}$ is the applied force.

Solution:

Newton's first law states that the time rate of change of the linear momentum is equal to the applied force. Let a body $\mathcal{B}$ have a motion $\boldsymbol{\varphi}$, and call the mapped body in the spatial configuration $\mathcal{S}$. The linear momentum for this body is,

$$\mathbf{l} = \int_B \rho \mathbf{v} \, d\mathbf{x} \,. \tag{13}$$

Its time rate of change is,

$$\begin{aligned} \dot{\mathbf{l}} &= \frac{D}{Dt} \int_S \rho \mathbf{v} \, d\mathbf{x} \\ &= \frac{D}{Dt} \int_B \rho \mathbf{v} J \, d\mathbf{X} \\ &= \frac{D}{Dt} \int_B \rho_0 \mathbf{v} \, d\mathbf{X} \\ &= \int_B \rho_o \dot{\mathbf{v}} \, d\mathbf{X} \\ &= \int_B \rho_o \mathbf{a} \, d\mathbf{X} \\ &= \int_B \rho J \mathbf{a} \, d\mathbf{X} \\ &= \int_S \rho \mathbf{a} \, d\mathbf{x} \,. \end{aligned} \tag{14}$$

The time derivative of the center of mass is,

$$
\begin{aligned}
\dot{\bar{\mathbf{x}}} &= \frac{D}{Dt}\frac{1}{m(S)}\int_S \rho\mathbf{x}\, d\mathbf{x} \\
&= \frac{1}{m(S)}\frac{D}{Dt}\int_B \rho\mathbf{x}J\, d\mathbf{X} \\
&= \frac{1}{m(S)}\frac{D}{Dt}\int_B \rho_0\mathbf{x}\, d\mathbf{X} \\
&= \frac{1}{m(S)}\int_B \rho_o\dot{\mathbf{x}}\, d\mathbf{X} \\
&= \frac{1}{m(S)}\int_B \rho_o\mathbf{v}\, d\mathbf{X} \\
&= \frac{1}{m(S)}\int_B \rho J\mathbf{v}\, d\mathbf{X} \\
&= \frac{1}{m(S)}\int_S \rho\mathbf{v}\, d\mathbf{x} \\
&= \frac{1}{m(S)}\mathbf{l}\,.
\end{aligned} \tag{15}
$$

Thus,

$$
\begin{aligned}
\dot{\mathbf{l}} &= \frac{D}{Dt}\left(m(S)\dot{\bar{\mathbf{x}}}\right) \\
&= m(\mathcal{S})\ddot{\bar{\mathbf{x}}}\,.
\end{aligned} \tag{16}
$$

From Newton's first law,

$$
\begin{aligned}
\dot{\mathbf{l}} &= \mathbf{f} \\
m(\mathcal{S})\ddot{\bar{\mathbf{x}}} &= \mathbf{f}
\end{aligned} \tag{17}
$$

thus yields the equation of motion for the whole body.

Problem:

Given a body with gravitational loading, show that the total sum of the reaction forces form the support point in the opposite direction of gravity with magnitude equal to the total mass times the gravitational acceleration.

Solution:

Let a body $\mathcal{S}$ be given in the spatial configuration. The gravitational acceleration is defined as $\mathbf{g}$, and the surface of the body with supports or applied traction is denoted as $\partial\mathcal{S}_t$. The balance of linear momentum for this body is,

$$
\begin{aligned}
0 &= \int_S \rho\mathbf{b}\, d\mathbf{x} + \int_{\partial S} \mathbf{t}\, da \\
&= \int_S \rho\mathbf{g}\, d\mathbf{x} + \int_{\partial\mathcal{S}_t} \mathbf{t}\, da \\
&= \int_S \rho\, d\mathbf{x}\mathbf{g} + \mathbf{R} \\
&= m(\mathcal{S})\mathbf{g} + \mathbf{R}\,.
\end{aligned} \tag{18}
$$

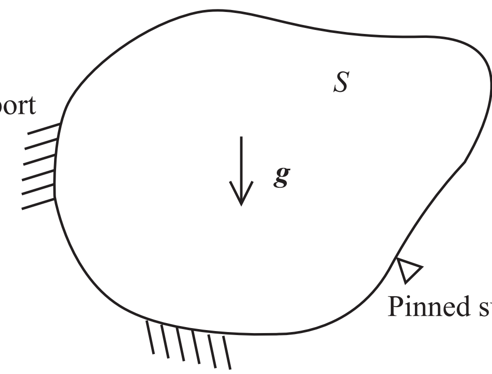

Figure 1: Configuration of the body under gravitational load

Here $\mathbf{R}$ is defined as the sum of the reaction forces on the surface $\mathcal{S}_t$. Thus,

$$\mathbf{R} = -m(\mathcal{S})\mathbf{g} \, . \tag{19}$$

Problem:

Show the the version of the principal of virtual work often used in basic mechanics.

1. The balance of linear momentum
2. The balance of angular momentum

hold if and only if,

1. Given any displacement field of the form,

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_0 + \mathbf{W}\mathbf{x} \tag{20}$$

where $\mathbf{u}_0$ is a rigid body displacement and $\mathbf{W}$ is a skew tensor which represents an infinitesimal rotation,

$$\int_S \rho(\mathbf{b} - \mathbf{a}) \cdot \mathbf{u} \, d\mathbf{x} + \int_{\partial S} \mathbf{t} \cdot \mathbf{u} \, d\mathbf{x} = 0 \tag{21}$$

holds.

Solution:

Before the proof is given, a brief explanation of why an infinitesimal rotation can be expressed as a skew tensor will be given. Assume the 2D case, where a rotation is expressed as,

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . \tag{22}$$

Assuming that $\theta << 1$,

$$\begin{aligned} \mathbf{R} &\approx \begin{bmatrix} 1 & -\theta \\ \theta & 1 \end{bmatrix} \\ &= \mathbf{1} + \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \\ &= \mathbf{1} + \mathbf{W}\,. \end{aligned} \tag{23}$$

It can be seen that for "infinitesimall small" rotations, $\mathbf{R}$ can be expressed as the sum of an identity plus a skew tensor. Given a motion of a rotation,

$$\begin{aligned} \mathbf{x} &= \mathbf{R}\mathbf{X} \\ &= (\mathbf{1} + \mathbf{W})\mathbf{X} \end{aligned} \tag{24}$$

and the definition of a displacement,

$$\begin{aligned} \mathbf{u} &= \mathbf{x} - \mathbf{X} \\ &= (\mathbf{1} + \mathbf{W})\mathbf{X} - \mathbf{X} \\ &= \mathbf{W}\mathbf{X} \end{aligned} \tag{25}$$

one can see that the displacement that arises from an infinitesimal rotation can be expressed by the skew tensor.

With this in mind, the proof of the theorem is given. Here only sufficiency is proved. The necessity is straightforward and closely resembles the proof for suffiency. Assume that given,

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_0 + \mathbf{W}\mathbf{x} \tag{26}$$

the equation,

$$\int_S \rho(\mathbf{b} - \mathbf{a}) \cdot \mathbf{u}\, d\mathbf{x} + \int_{\partial S} \mathbf{t} \cdot \mathbf{u}\, d\mathbf{x} = 0 \tag{27}$$

holds. Inserting the expression for the displacement yields,

$$\begin{aligned} 0 &= \int_S \rho(\mathbf{b} - \mathbf{a}) \cdot \mathbf{u}_0\, d\mathbf{x} + \int_S \rho(\mathbf{b} - \mathbf{a}) \cdot \mathbf{W}\mathbf{x}\, d\mathbf{x} + \int_{\partial S} \mathbf{t} \cdot \mathbf{u}_0\, d\mathbf{x} + \int_{\partial S} \mathbf{t} \cdot \mathbf{W}\mathbf{x}\, d\mathbf{x} \\ &= \int_S \rho(\mathbf{b} - \mathbf{a})\, d\mathbf{x} \cdot \mathbf{u}_0 + \int_{\partial S} \mathbf{t}\, d\mathbf{x} \cdot \mathbf{u}_0 + \int_S \rho(\mathbf{b} - \mathbf{a}) \cdot (\boldsymbol{\omega} \times \mathbf{x})\, d\mathbf{x} + \int_{\partial S} \mathbf{t} \cdot (\boldsymbol{\omega} \times \mathbf{x})\, d\mathbf{x} \\ &= \left(\int_S \rho(\mathbf{b} - \mathbf{a})\, d\mathbf{x} + \int_{\partial S} \mathbf{t}\, d\mathbf{x} \right) \cdot \mathbf{u}_0 + \int_S \boldsymbol{\omega} \cdot (\mathbf{x} \times \rho(\mathbf{b} - \mathbf{a}))\, d\mathbf{x} + \int_{\partial S} \boldsymbol{\omega} \cdot (\mathbf{x} \times \mathbf{t})\, d\mathbf{x} \\ &= \left(\int_S \rho(\mathbf{b} - \mathbf{a})\, d\mathbf{x} + \int_{\partial S} \mathbf{t}\, d\mathbf{x} \right) \cdot \mathbf{u}_0 + \boldsymbol{\omega} \cdot \left(\int_S \mathbf{x} \times \rho(\mathbf{b} - \mathbf{a})\, d\mathbf{x} + \int_{\partial S} \mathbf{x} \times \mathbf{t}\, d\mathbf{x} \right) \end{aligned} \tag{28}$$

Here the relation that every skew tensor has a represention as its axial vector has been used. This has been followed with the rearrangement of the cross product,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})\,. \tag{29}$$

Since the selection of $\mathbf{u}_0$ and $\mathbf{W}$(or $\boldsymbol{\omega}$) is arbitrary,

$$\int_S \rho(\mathbf{b} - \mathbf{a})\, d\mathbf{x} + \int_{\partial S} \mathbf{t}\, d\mathbf{x} = 0 \tag{30}$$

$$\int_S \mathbf{x} \times \rho(\mathbf{b} - \mathbf{a})\, d\mathbf{x} + \int_{\partial S} \mathbf{x} \times \mathbf{t}\, d\mathbf{x} = 0 \tag{31}$$

and the balance of linear and angular momentum are implied.

Problem:

Show the continuity equation,

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0\,. \tag{32}$$

Solution:

One obtaines the local form of mass conservation,

$$\rho_0 = \rho J \tag{33}$$

from the integral form through the localization theorem. ρ_0 depends only on the material coordinates and is independent of time. By taking a material time derivative of both sides of the equation one obtains,

$$\begin{aligned}
\dot{\rho}_0 &= \dot{\overline{\rho J}} \\
0 &= \dot{\rho} J + \rho \dot{J} \\
0 &= \dot{\rho} J + \rho \frac{\partial J}{\partial \mathbf{F}} : \dot{\mathbf{F}} \\
&= \dot{\rho} J + \rho J \mathbf{F}^{-T} : \dot{\mathbf{F}} \\
&= \dot{\rho} J + \rho J \mathbf{1} : \dot{\mathbf{F}} \mathbf{F}^{-1} \\
&= J \left(\dot{\rho} + \rho \operatorname{tr}(\mathbf{l}) \right) \\
&= J \left(\dot{\rho} + \rho \operatorname{div}(\mathbf{v}) \right)\,.
\end{aligned} \tag{34}$$

Since $J > 0$, the two sides of the equation above are divided to obtain,

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0\,. \tag{35}$$

Problem:

Given a rigid body motion,

$$\mathbf{x} = \mathbf{R}(t) \left(\mathbf{X} - \mathbf{0} \right) + \mathbf{c}(t) \tag{36}$$

where $\mathbf{R}(t)$ an orthogonal tensor, show that the linear momentum $\mathbf{m}$ and angular momentum $\mathbf{h}$ is given as,

$$\mathbf{m} = m(P_t) \overline{\mathbf{v}} \tag{37}$$

$$\mathbf{h} = m(P_t) \overline{\mathbf{x}} \times \overline{\mathbf{v}} + \overline{\mathbb{J}} \omega. \tag{38}$$

Here we have defined,

$$m(P_t) = \int_{P_t} \rho \, d\mathbf{x} \tag{39}$$

$$\mathbf{\Omega} = \dot{\mathbf{R}} \mathbf{R}^T \tag{40}$$

$$\omega \times \mathbf{a} = \mathbf{\Omega} \mathbf{a} \tag{41}$$

$$\overline{\mathbf{x}} = \frac{1}{m(P_t)} \int_{P_t} \rho \mathbf{x} \, d\mathbf{x} \tag{42}$$

$$\overline{\mathbf{v}} = \omega \times (\overline{\mathbf{x}} - \mathbf{c}) + \dot{\mathbf{c}} \tag{43}$$

$$\mathbb{J} = \int_{P_t} \left[(\mathbf{x} \cdot \mathbf{x}) \mathbf{1} - \mathbf{x} \otimes \mathbf{x} \right] \, d\mathbf{x} \tag{44}$$

$$\mathbf{z} = \mathbf{x} - \overline{\mathbf{x}} \tag{45}$$

$$\overline{\mathbb{J}} = \int_{P_t} \left[(\mathbf{z} \cdot \mathbf{z}) \mathbf{1} - \mathbf{z} \otimes \mathbf{z} \right] \, d\mathbf{x} \tag{46}$$

Solution:

To compute the linear momentum the spatial velocity must first be calculated.

$$\mathbf{x} = \mathbf{R}\mathbf{X} + \mathbf{c} \tag{47}$$

implies,

$$\mathbf{X} = \mathbf{R}^T (\mathbf{x} - \mathbf{c}) \ . \tag{48}$$

From this the material velocity becomes,

$$\mathbf{V} = \dot{\mathbf{R}}\mathbf{X} + \dot{\mathbf{c}} \tag{49}$$

and thus the spatial velocity is,

$$\begin{aligned} \mathbf{v} &= \dot{\mathbf{R}}\mathbf{R}^T (\mathbf{x} - \mathbf{c}) + \dot{\mathbf{c}} & (50)\\ &= \boldsymbol{\Omega} (\mathbf{x} - \mathbf{c}) + \dot{\mathbf{c}} & (51)\\ &= \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{c}) + \dot{\mathbf{c}} \ . & (52) \end{aligned}$$

The linear momentum of the body is obtained by the expression,

$$\begin{aligned} \mathbf{m} &= \int_{P_t} \rho \mathbf{v} \, d\mathbf{x} \\ &= \int_{P_t} \rho \left[\omega \times (\mathbf{x} - \mathbf{c}) + \dot{\mathbf{c}}\right] \, d\mathbf{x} \\ &= \int_{P_t} \rho\omega \times (\mathbf{x} - \mathbf{c}) + \rho\dot{\mathbf{c}} \, d\mathbf{x} \\ &= \omega \times \int_{P_t} \rho (\mathbf{x} - \mathbf{c}) \, d\mathbf{x} + \dot{\mathbf{c}} \int_{P_t} \rho \, d\mathbf{x} \\ &= \omega \times m(P_t) (\overline{\mathbf{x}} - \mathbf{c}) + \dot{\mathbf{c}} m(P_t) \\ &= m(P_t) \left[\omega \times (\overline{\mathbf{x}} - \mathbf{c}) + \dot{\mathbf{c}}\right] \\ &= m(P_t) \overline{\mathbf{v}} \ . & (53) \end{aligned}$$

$\overline{\mathbf{v}}$ is the velocity of the mass center. The angular momentum of the body is obtained by the expression,

$$\begin{aligned} \mathbf{h} &= \int_{P_t} \mathbf{x} \times \rho \mathbf{v} \, d\mathbf{x} \\ &= \int_{P_t} \mathbf{x} \times \rho \left[\omega \times (\mathbf{x} - \mathbf{c}) + \dot{\mathbf{c}}\right] \, d\mathbf{x} \\ &= \int_{P_t} \rho \mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{x}) - \rho \mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{c}) + \rho \mathbf{x} \times \dot{\mathbf{c}} \, d\mathbf{x} \ . & (54) \end{aligned}$$

Using the identity,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \, \mathbf{c} \tag{55}$$

one can see that,

$$\begin{aligned}
&\int_{P_t} \rho \mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{x}) \ d\mathbf{x} \\
&\quad = \int_{P_t} \rho \left[(\mathbf{x} \cdot \mathbf{x}) \boldsymbol{\omega} - (\mathbf{x} \cdot \boldsymbol{\omega}) \mathbf{x} \right] \ d\mathbf{x} \\
&\quad = \int_{P_t} \rho \left[(\mathbf{x} \cdot \mathbf{x}) \mathbf{1} - \mathbf{x} \otimes \mathbf{x} \right] \boldsymbol{\omega} \ d\mathbf{x} \\
&\quad = \int_{P_t} \rho \left[(\mathbf{x} \cdot \mathbf{x}) \mathbf{1} - \mathbf{x} \otimes \mathbf{x} \right] \ d\mathbf{x} \boldsymbol{\omega} \\
&\quad = \mathbb{J} \omega \ .
\end{aligned} \tag{56}$$

The value $\mathbb{J}$ obtained is the moment of inertia tensor,

$$\begin{aligned}
\mathbb{J} &= \int_{P_t} \rho \begin{bmatrix} x_2^2 + x_3^2 & -x_1 x_2 & -x_1 x_3 \\ -x_2 x_1 & x_3^2 + x_1^2 & -x_2 x_3 \\ -x_3 x_1 & -x_3 x_2 & x_1^2 + x_2^2 \end{bmatrix} d\mathbf{x} \\
&= \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} .
\end{aligned} \tag{57}$$

This inertia tensor is given with respect to the origin. One can obtain an inertial tensor with respect to the center of mass $\overline{\mathbf{x}}$. This will be noted as $\overline{\mathbb{J}}$. The relation with $\mathbb{J}$ can be obtained as follows.

$$\begin{aligned}
\overline{\mathbb{J}} &= \int_{P_t} \rho \left[(\mathbf{z} \cdot \mathbf{z}) \mathbf{1} - \mathbf{z} \otimes \mathbf{z} \right] \ d\mathbf{x} \\
&= \int_{P_t} \rho \left[((\mathbf{x} - \overline{\mathbf{x}}) \cdot (\mathbf{x} - \overline{\mathbf{x}})) \mathbf{1} - (\mathbf{x} - \overline{\mathbf{x}}) \otimes (\mathbf{x} - \overline{\mathbf{x}}) \right] \ d\mathbf{x} \\
&= \int_{P_t} \rho \left[(\mathbf{x} \cdot \mathbf{x} - 2\overline{\mathbf{x}} \cdot \mathbf{x} + \overline{\mathbf{x}} \cdot \overline{\mathbf{x}}) \mathbf{1} \left(\mathbf{x} \otimes \mathbf{x} - \mathbf{x} \otimes \overline{\mathbf{x}} - \overline{\mathbf{x}} \otimes \mathbf{x} + \overline{\mathbf{x}} \otimes \overline{\mathbf{x}} \right) \right] \ d\mathbf{x} \\
&= \int_{P_t} \rho \left[\mathbf{x} \cdot \mathbf{x} \mathbf{1} - \mathbf{x} \otimes \mathbf{x} \right] \ d\mathbf{x} + \int_{P_t} \rho \left[(-2\overline{\mathbf{x}} \cdot \mathbf{x} + \overline{\mathbf{x}} \cdot \overline{\mathbf{x}}) \mathbf{1} - (-\mathbf{x} \otimes \overline{\mathbf{x}} - \overline{\mathbf{x}} \otimes \mathbf{x} + \overline{\mathbf{x}} \otimes \overline{\mathbf{x}}) \right] \ d\mathbf{x} \\
&= \mathbb{J} + (-2m(P_t) \overline{\mathbf{x}} \cdot \overline{\mathbf{x}} + m(P_t) \overline{\mathbf{x}} \cdot \overline{\mathbf{x}}) \mathbf{1} - (-m(P_t) \overline{\mathbf{x}} \otimes \overline{\mathbf{x}} - m(P_t) \overline{\mathbf{x}} \otimes \overline{\mathbf{x}} + m(P_t) \overline{\mathbf{x}} \otimes \overline{\mathbf{x}}) \\
&= \mathbb{J} - m(P_t) \left(\overline{\mathbf{x}} \cdot \overline{\mathbf{x}} \mathbf{1} - \overline{\mathbf{x}} \otimes \overline{\mathbf{x}} \right) \\
&= \mathbb{J} - \mathbb{J}_{\overline{\mathbf{x}}}
\end{aligned} \tag{58}$$

where,

$$\begin{aligned}
\mathbb{J}_{\overline{\mathbf{x}}} &= m(P_t) \left(\overline{\mathbf{x}} \cdot \overline{\mathbf{x}} \mathbf{1} - \overline{\mathbf{x}} \otimes \overline{\mathbf{x}} \right) \\
&= \int_{P_t} \rho \left[(\overline{\mathbf{x}} \cdot \overline{\mathbf{x}}) \mathbf{1} - \overline{\mathbf{x}} \otimes \overline{\mathbf{x}} \right] \ d\mathbf{x}
\end{aligned} \tag{59}$$

is the moment of inertia of the center of mass. This expression is what one would expect. Inserting the expressions for $\mathbb{J}$ in eqn. (54),

$$\begin{aligned}
\mathbf{h} &= \mathbb{J}\boldsymbol{\omega} + \int_{P_t} \rho\mathbf{x} \times (\dot{\mathbf{c}} - \boldsymbol{\omega} \times \mathbf{c})\ d\mathbf{x} \\
&= \mathbb{J}\boldsymbol{\omega} + \int_{P_t} \rho\mathbf{x}\, d\mathbf{x} \times (\dot{\mathbf{c}} - \boldsymbol{\omega} \times \mathbf{c}) \\
&= \mathbb{J}\boldsymbol{\omega} + m(P_t)\, [\overline{\mathbf{x}} \times (\dot{\mathbf{c}} - \boldsymbol{\omega} \times \mathbf{c})] \\
&= \overline{\mathbb{J}}\boldsymbol{\omega} + \mathbb{J}_{\overline{\mathbf{x}}}\boldsymbol{\omega} + m(P_t)\, [\overline{\mathbf{x}} \times (\dot{\mathbf{c}} - \boldsymbol{\omega} \times \mathbf{c})] \\
&= \overline{\mathbb{J}}\boldsymbol{\omega} + \mathbb{J}_{\overline{\mathbf{x}}}\boldsymbol{\omega} + m(P_t)\overline{\mathbf{x}} \times \overline{\mathbf{v}} - m(P_t)\overline{\mathbf{x}} \times (\boldsymbol{\omega} \times \overline{\mathbf{x}}) \qquad (60)
\end{aligned}$$

Since,

$$\begin{aligned}
\mathbb{J}_{\overline{\mathbf{x}}}\boldsymbol{\omega} &= m(P_t)\,(\overline{\mathbf{x}} \cdot \overline{\mathbf{x}}\mathbf{1} - \overline{\mathbf{x}} \otimes \overline{\mathbf{x}})\,\boldsymbol{\omega} \\
&= m(P_t)\,(\overline{\mathbf{x}} \cdot \overline{\mathbf{x}})\,\boldsymbol{\omega} - (\overline{\mathbf{x}} \cdot \boldsymbol{\omega})\,\overline{\mathbf{x}} \\
&= m(P_t)\overline{\mathbf{x}} \times (\boldsymbol{\omega} \times \overline{\mathbf{x}})\ , \qquad (61)
\end{aligned}$$

the expression for $\mathbf{h}$ becomes,

$$\mathbf{h} = \overline{\mathbb{J}}\boldsymbol{\omega} + m(P_t)\overline{\mathbf{x}} \times \overline{\mathbf{v}}\ . \qquad (62)$$

Problem:

Given the traction $\mathbf{t}$ on a surface and the surface normal $\mathbf{n}$, decompose $\mathbf{t}$ into its contribution parallel to the normal $\mathbf{t}_n$ and orthogonal to the normal $\mathbf{t}_s$.

$$\mathbf{t} = \mathbf{t}_n + \mathbf{t}_s \tag{63}$$

Solution:

The contribution parallel to the normal can be obtained by,

$$\begin{aligned} \mathbf{t}_n &= (\mathbf{t} \cdot \mathbf{n})\, \mathbf{n} && (64)\\ &= (\mathbf{n} \otimes \mathbf{n})\, \mathbf{t} && (65)\\ &= \mathbb{P}_n^{||} \mathbf{n}\,. && (66) \end{aligned}$$

This is the force acting normal to the surface. As one can see, $\mathbf{t}_n$ can be obtained by using the projection operator $\mathbb{P}_n^{||}$, which projects the traction vector $\mathbf{t}$ onto $\mathbf{n}$. The contribution orthogonal to the normal can be obtained by,

$$\begin{aligned} \mathbf{t}_s &= \mathbf{t} - \mathbf{t}_n && (67)\\ &= \mathbf{t} - \mathbb{P}_n^{||} \mathbf{t} && (68)\\ &= \left(\mathbf{1} - \mathbb{P}_n^{||}\right) \mathbf{t} && (69)\\ &= \mathbb{P}_n^{\perp} \mathbf{t}\,. && (70) \end{aligned}$$

This is the force acting parallel to the surface, often called a shear force. As one can see, $\mathbf{t}_n$ can be obtained by using the projection operator $\mathbb{P}_n^{\perp}$, which projects the traction vector $\mathbf{t}$ onto the plane normal to $\mathbf{n}$.

Problem:

Prove necessity in Cauchy's theorem. Given a continuous traction field $\mathbf{t}(\mathbf{x}, \mathbf{n})$ and body force field (per unit mass) $\mathbf{b}(\mathbf{x})$, show that,

1. Global linear momentum balance,
2. Global angular momentum balance,

implies the existence of a tensor $\boldsymbol{\sigma}(\mathbf{x})$ such that,

1. $\mathbf{t}(\mathbf{x}, \mathbf{n}) = \boldsymbol{\sigma}^T(\mathbf{x})\mathbf{n}$,
2. $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$,
3. $\text{div}(\boldsymbol{\sigma}^T) + \rho\mathbf{b} = \rho\mathbf{a}$.

Solution:

Step 1: Show that $\mathbf{t}(\mathbf{x}, -\mathbf{n}) = -\mathbf{t}(\mathbf{x}, \mathbf{n})$

Take a region P_t within the body $\mathcal{S}$ in the shape of a disk with height d and radius R. The surface normal to the top of the disk is noted $\mathbf{n}$ and the bottom $-\mathbf{n}$. The disk is assumed infinitesimally small such that it is possible to assume $\mathbf{t},\ \mathbf{b},\ ,\mathbf{a}$, and ρ constant in P_t. (To properly do this one can use a mean value argument and say that,

$$\mathbf{b}(\mathbf{x}_0) = \frac{1}{\text{vol}(P_t)} \int_{P_t} \mathbf{b}(\mathbf{x})\, d\mathbf{x} \tag{71}$$

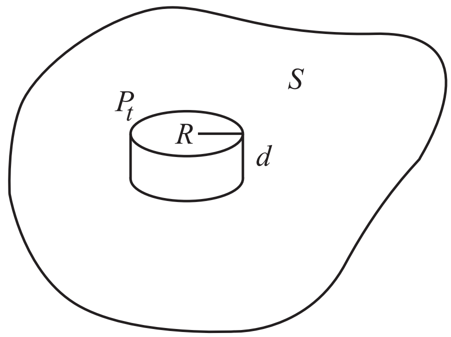

Figure 2: Configuration of the cylinder inside of the body

for some $\mathbf{x}_0$, but these are technicalities. One can refer to Gurtin page 101 for further details). In doing so we also suppress the dependence on $\mathbf{x}$ for clarity. Applying global linear momentum balance to this region yields,

$$\begin{aligned}
\partial P_t &= \partial P_{\text{top}} \cup \partial P_{\text{bottom}} \cup \partial P_{\text{side}} && (72)\\
0 &= \int_{\partial P_t} \mathbf{t}\, da + \int_{P_t} \rho(\mathbf{b}-\mathbf{a})\, d\mathbf{x} \\
&= \mathbf{t}(\mathbf{n})\pi R^2 + \mathbf{t}(-\mathbf{n})\pi R^2 + \int_{\text{side}} \mathbf{t}\, da + \rho(\mathbf{b}-\mathbf{a})\pi R^2 d \\
&= \mathbf{t}(\mathbf{n})\pi R^2 + \mathbf{t}(-\mathbf{n})\pi R^2 + \int_{\text{side}} \mathbf{t} R d\, d\theta + \rho(\mathbf{b}-\mathbf{a})\pi R^2 d \\
&= \mathbf{t}(\mathbf{n})\pi R^2 + \mathbf{t}(-\mathbf{n})\pi R^2 + d\left(\int_{\text{side}} \mathbf{t} R\, d\theta + \rho(\mathbf{b}-\mathbf{a})\pi R^2\right) \\
&= \mathbf{t}(\mathbf{n})\pi R^2 + \mathbf{t}(-\mathbf{n})\pi R^2 + d\left(\int_{\text{side}} \mathbf{t} R\, d\theta + \rho(\mathbf{b}-\mathbf{a})\pi R^2\right) . && (73)
\end{aligned}$$

Using the continuity of the fields, the limit of $d \to 0$ allows us to assert that,

$$0 = \mathbf{t}(\mathbf{n})\pi R^2 + \mathbf{t}(-\mathbf{n})\pi R^2$$

and thus,

$$0 = \mathbf{t}(\mathbf{n})\pi + \mathbf{t}(-\mathbf{n})\pi . \qquad (74)$$

From this one can see that Newtons 3rd law of action reaction is not really a law.

Step 2: Show that $\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}^T \mathbf{n}$

Take a region P_t within the body $\mathcal{S}$ in the shape of a tetrahedron with height h and base area A. Set an orthogonal coordinate system $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ such that the three faces are aligned with this. The surface normal to the base is

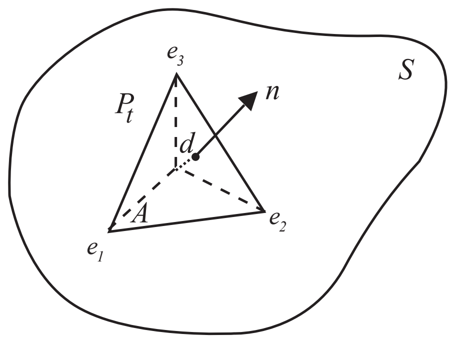

Figure 3: Configuration of the tetrahedron inside the body

denoted $\mathbf{n}$. The tetrahedron is again assumed infinitesimally small such that it is possible to assume $\mathbf{t},\ \mathbf{b},\ ,\mathbf{a}$, and ρ constant in P_t for the same reasons as given previously in Step 1. In doing so we also suppress the dependence on $\mathbf{x}$ for clarity. Applying global linear momentum balance to this region yields,

$$\begin{aligned}
\partial P_t &= \partial P_{\text{base}} \cup \partial P_{\text{yz}} \cup \partial P_{\text{zx}} \cup \partial P_{\text{xy}} && (75)\\
0 &= \int_{\partial P_t} \mathbf{t}\, da + \int_{P_t} \rho(\mathbf{b}-\mathbf{a})\, d\mathbf{x} \\
&= \mathbf{t}(\mathbf{n})A\mathbf{t}(-\mathbf{e}_1)A_{\text{yz}} + \mathbf{t}(-\mathbf{e}_2)A_{\text{zx}} + \mathbf{t}(-\mathbf{e}_3)A_{\text{xy}} + \rho(\mathbf{b}-\mathbf{a})\frac{1}{3}Ah \\
&= \mathbf{t}(\mathbf{n})A\mathbf{t}(-\mathbf{e}_1)A(\mathbf{n}\cdot\mathbf{e}_1) + \mathbf{t}(-\mathbf{e}_2)A(\mathbf{n}\cdot\mathbf{e}_2) + \mathbf{t}(-\mathbf{e}_3)A(\mathbf{n}\cdot\mathbf{e}_3) + \rho(\mathbf{b}-\mathbf{a})\frac{1}{3}Ah\,. && (76)
\end{aligned}$$

First dividing by the area A and then using the continuity of the fields, the limit of $h \to 0$ allows us to assert that,

$$0 = \mathbf{t}(\mathbf{n})\mathbf{t}(-\mathbf{e}_1)(\mathbf{n}\cdot\mathbf{e}_1) + \mathbf{t}(-\mathbf{e}_2)(\mathbf{n}\cdot\mathbf{e}_2) + \mathbf{t}(-\mathbf{e}_3)(\mathbf{n}\cdot\mathbf{e}_3)$$

and thus using the result from Step 1,

$$\begin{aligned}
\mathbf{t}(\mathbf{n}) &= -\mathbf{t}(-\mathbf{e}_i)n_i \\
&= \mathbf{t}(\mathbf{e}_i)n_i \\
\mathbf{t}(n_i\mathbf{e}_i) &= \mathbf{t}(\mathbf{e}_i)n_i\,. && (77)
\end{aligned}$$

One can see that $\mathbf{t}$ is a linear operator acting on $\mathbf{n}$. Denote this operation as $\boldsymbol{\sigma}^T$ and define,

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}^T\mathbf{n}\,. \qquad (78)$$

This shows the existence of a stress tensor, which is named the Cauchy stress tensor. This tensor is defined in the spatial configuration, and the traction vector obtained is given in terms of force per unit "deformed area". Thus it is often called the true stress in engineering and distinguished from the nominal stress which is defined per unit "undeformed area". The traction in the $\mathbf{e}_1$ is obtained from the definition as,

$$\begin{aligned} t(\mathbf{e}_1) &= \sigma_{ij}\mathbf{e}_j \otimes \mathbf{e}_i\mathbf{e}_1 \\ &= \sigma_{1j}\mathbf{e}_j \, . \end{aligned} \tag{79}$$

This reveals that the definition given of the tensor σ in terms of its transpose , results in the first index of σ defining the face of on which the stress is being observed. Thus σ_{12} is the shear force acting on face 1 in the direction of 2. Had the definition of σ be given as,

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n} \, . \tag{80}$$

as some textbooks such as Holzapfel and Gurtin, the second index would define the face. In this case, σ_{12} would be the shear force acting on face 2 in the direction of 1.

Step 3: Show that $\text{div}(\boldsymbol{\sigma}^T) + \rho\mathbf{b} = \rho\mathbf{a}$.

Take an arbitrary region P_t within the body $\mathcal{S}$. The global balance of linear momentum supplies the relation,

$$\int_{P_t} \rho(\mathbf{b} - \mathbf{a}) \, d\mathbf{x} + \int_{\partial P_t} \mathbf{t} \, da = 0 \, . \tag{81}$$

Inserting the definition for stress σ obtained in Step 2,

$$\int_{P_t} \rho(\mathbf{b} - \mathbf{a}) \, d\mathbf{x} + \int_{\partial P_t} \boldsymbol{\sigma}^T\mathbf{n} \, da = 0 \tag{82}$$

and an application of the Divergence theorem transforming the surface integral to a volume integral yields,

$$\begin{aligned} \int_{P_t} \rho(\mathbf{b} - \mathbf{a}) \, d\mathbf{x} + \int_{P_t} \text{div}(\boldsymbol{\sigma}^T) \, d\mathbf{x} &= 0 \\ \int_{P_t} \rho(\mathbf{b} - \mathbf{a}) + \text{div}(\boldsymbol{\sigma}^T) \, d\mathbf{x} &= \end{aligned} \tag{83}$$

Since the region P_t is arbitrary, an application of the Localization theorem results in the equation,

$$\text{div}(\boldsymbol{\sigma}^T) \, d\mathbf{x}\rho\mathbf{b} = \rho\mathbf{a} \, . \tag{84}$$

Step 4: Show that $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$.

Take an arbitrary region P_t within the body $\mathcal{S}$. The global balance of angular momentum supplies the relation,

$$\int_{P_t} \mathbf{x} \times \rho(\mathbf{b} - \mathbf{a}) \, d\mathbf{x} + \int_{\partial P_t} \mathbf{x} \times \mathbf{t} \, da = 0 \, . \tag{85}$$

By insertion of the definition for stress σ obtained in Step 2,

$$\int_{P_t} \mathbf{x} \times \rho(\mathbf{b} - \mathbf{a}) \, d\mathbf{x} + \int_{\partial P_t} \mathbf{x} \times (\boldsymbol{\sigma}^T\mathbf{n}) \, da = 0 \, . \tag{86}$$

For the next step in application of the Divergence theorem, index notation is more suitable for clarity and will be employed.

$$
\begin{aligned}
0 &= \int_{P_t} \mathbf{x} \times \rho(\mathbf{b}-\mathbf{a})\, d\mathbf{x} + \int_{\partial P_t} x_i (\boldsymbol{\sigma}^T)_{kj} n_j \mathbf{e}_l \varepsilon_{ikl}\, da \\
&= \int_{P_t} \mathbf{x} \times \rho(\mathbf{b}-\mathbf{a})\, d\mathbf{x} + \int_{P_t} \left[x_i (\boldsymbol{\sigma}^T)_{kj} \mathbf{e}_l \varepsilon_{ikl}\right]_{,j}\, d\mathbf{x} \\
&= \int_{P_t} \mathbf{x} \times \rho(\mathbf{b}-\mathbf{a})\, d\mathbf{x} + \int_{P_t} x_{i,j} (\boldsymbol{\sigma}^T)_{kj} \mathbf{e}_l \varepsilon_{ikl} + x_i (\boldsymbol{\sigma}^T)_{kj,j} \mathbf{e}_l \varepsilon_{ikl}\, d\mathbf{x} \\
&= \int_{P_t} \mathbf{x} \times \rho(\mathbf{b}-\mathbf{a})\, d\mathbf{x} + \int_{P_t} \delta_{ij} (\boldsymbol{\sigma}^T)_{kj} \mathbf{e}_l \varepsilon_{ikl} + x_i (\mathrm{div}\boldsymbol{\sigma}^T)_k \mathbf{e}_l \varepsilon_{ikl}\, d\mathbf{x} \\
&= \int_{P_t} \mathbf{x} \times \rho(\mathbf{b}-\mathbf{a})\, d\mathbf{x} + \int_{P_t} (\boldsymbol{\sigma}^T)_{kj} \mathbf{e}_l \varepsilon_{jkl} + \mathbf{x} \times (\mathrm{div}\boldsymbol{\sigma}^T)\, d\mathbf{x} \\
&= \int_{P_t} (\boldsymbol{\sigma}^T)_{kj} \mathbf{e}_l \varepsilon_{jkl}\, d\mathbf{x}\,.
\end{aligned} \tag{87}
$$

In the last step, the equilibrium equation obtained in Step 3 has been applied. Since the region P_t is arbitrary, an application of the Localization theorem results in the equation,

$$(\boldsymbol{\sigma}^T)_{kj} \mathbf{e}_l \varepsilon_{jkl} = 0\,. \tag{88}$$

Linear independency of the basis vectors $\mathbf{e}_i$ implies that,

$$\sigma_{jk} \varepsilon_{jkl} = 0 \tag{89}$$

for all l. This yields three equations,

$$
\begin{aligned}
\sigma_{23} - \sigma 32 &= 0 \qquad (90)\\
\sigma_{31} - \sigma 13 &= 0 \qquad (91)\\
\sigma_{12} - \sigma 21 &= 0 \qquad (92)
\end{aligned}
$$

which shows that the Cauchy stress tensor $\boldsymbol{\sigma}$ is symmetric.

Problem:

In a polar continuum, there exist distributed resultant couples,

1. Body couples or body torques $\mathbf{c}$, given per unit volume or mass,
2. Coupled traction vector or contact torque $\mathbf{m}$.

In this case the balance of angular momentum of the body is modified to include these effect,

$$\int_{P_t} \mathbf{x} \times \rho(\mathbf{b}-\mathbf{a}) + \mathbf{c}\, d\mathbf{x} + \int_{\partial P_t} \mathbf{x} \times \mathbf{t} + \mathbf{m}\, da = 0\,. \tag{93}$$

Using an argument similar to the tetrahedron argument in the proof of Cauchy's theorem, prove the existence of a couple stress tensor $\boldsymbol{\mu}$,

$$\boldsymbol{\mu}(\mathbf{x}, \mathbf{n}) = \boldsymbol{\mu}^T(\mathbf{x})\mathbf{n}\,. \tag{94}$$

Show that in this case Cauchy's stress tensor is not symmetric.

Solution:

Step 1: Simplify the expression for balance of angular momentum using balance of linear momentum.

The balance of linear momentum is not altered in the presence of resultant couples and thus the proof of the existence of Cauchy's stress tensor $\boldsymbol{\sigma}$ as well as the equilibrium equation does not change,

$$\text{div}(\boldsymbol{\sigma}^T) + \rho(\mathbf{b} - \mathbf{a}) \quad = \quad 0\,. \tag{95}$$

Let us define the permutation as a 3rd-order tensor,

$$\mathcal{E} \quad = \quad \varepsilon_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \tag{96}$$

and its double contraction with a 2nd-order tensor $\boldsymbol{\sigma}$ as,

$$\mathcal{E} : \boldsymbol{\sigma} \quad = \quad \varepsilon_{ijk}\sigma_{jk}\mathbf{e}_l\,. \tag{97}$$

Recall from the proof of the symmetry of Cauchy's stress tensor that,

$$\begin{aligned}\int_{\partial P_t} \mathbf{x} \times \mathbf{t} da &= \int_{\partial P_t} \mathbf{x} \times (\boldsymbol{\sigma}^T\mathbf{n}) da \\ &= \int_{P_t} \mathbf{x} \times \text{div}(\boldsymbol{\sigma}^T) + \mathcal{E} : \boldsymbol{\sigma}\; d\mathbf{x}\,.\end{aligned} \tag{98}$$

Inserting this relation into the modifed balance of angular momentum yields,

$$\int_{P_t} \mathbf{c} + \mathcal{E} : \boldsymbol{\sigma}\; d\mathbf{x} + \int_{\partial P_t} \mathbf{m}\; da = 0\,. \tag{99}$$

Step 2: Show that $\mathbf{m}(\mathbf{n}) = \boldsymbol{\mu}^T\mathbf{n}$

Take a region P_t within the body $\mathcal{S}$ in the shape of a tetrahedron with height h and base area A. Set an orthogonal coordinate system $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ such that the three faces are aligned with this. The surface normal to the base is denoted $\mathbf{n}$. The tetrahedron is again assumed infinitesimally small such that it is possible to assume $\mathbf{t},\ \mathbf{b},\ ,\mathbf{a},\ \mathbf{c},\ \mathbf{m}$ and ρ constant in P_t. In doing so we also suppress the dependence on $\mathbf{x}$ for clarity. Applying global angular momentum balance to this region yields,

$$\partial P_t \quad = \quad \partial P_{\text{base}} \cup \partial P_{\text{yz}} \cap \partial P_{\text{zx}} \cup \partial P_{\text{xy}} \tag{100}$$

$$\begin{aligned}0 &= \int_{\partial P_t} \mathbf{m}\; da + \int_{P_t} \mathbf{c}\mathcal{E} : \boldsymbol{\sigma}\; d\mathbf{x} \\ &= \mathbf{m}(\mathbf{n})A\mathbf{m}(-\mathbf{e}_1)A_{\text{yz}} + \mathbf{m}(-\mathbf{e}_2)A_{\text{zx}} + \mathbf{m}(-\mathbf{e}_3)A_{\text{xy}} + (\mathbf{c} + \mathcal{E} : \boldsymbol{\sigma})\frac{1}{3}Ah \\ &= \mathbf{m}(\mathbf{n})A\mathbf{m}(-\mathbf{e}_1)A(\mathbf{n}\cdot\mathbf{e}_1) + \mathbf{m}(-\mathbf{e}_2)A(\mathbf{n}\cdot\mathbf{e}_2) + \mathbf{m}(-\mathbf{e}_3)A(\mathbf{n}\cdot\mathbf{e}_3) + (\mathbf{c} + \mathcal{E} : \boldsymbol{\sigma})\frac{1}{3}Ah\,.\end{aligned} \tag{101}$$

Using the continuity of the fields, the limit of $h \to 0$ allows us to assert that,

$$0 \quad = \quad \mathbf{m}(\mathbf{n})A\mathbf{m}(-\mathbf{e}_1)A(\mathbf{n}\cdot\mathbf{e}_1) + \mathbf{m}(-\mathbf{e}_2)A(\mathbf{n}\cdot\mathbf{e}_2) + \mathbf{m}(-\mathbf{e}_3)A(\mathbf{n}\cdot\mathbf{e}_3)$$

and thus,

$$\begin{aligned}\mathbf{m}(\mathbf{n}) &= -\mathbf{m}(-\mathbf{e}_i)n_i \\ &= \mathbf{m}(\mathbf{e}_i)n_i \\ \mathbf{m}(n_i\mathbf{e}_i) &= \mathbf{m}(\mathbf{e}_i)n_i\,.\end{aligned} \tag{102}$$

Here we have used the property of $\mathbf{m}(-\mathbf{m}) = -\mathbf{m}(\mathbf{n})$ which can be shown similarly to the proof for Cauchy's stress tensor. Since this is trivial, it is omitted here. One can see that $\mathbf{m}$ is a linear operator acting on $\mathbf{n}$. Denote this operation as $\boldsymbol{\mu}^T$ and define,

$$\mathbf{m}(\mathbf{n}) \quad = \quad \boldsymbol{\mu}^T \mathbf{n} \, . \tag{103}$$

This shows the existence of a couple stress tensor.

Step 3: Show that Cauchy's stress tensor is not necessarily symmetric.

Inserting the result from Step 2 into the equation from Step 1 yields,

$$\begin{aligned} 0 &= \int_{P_t} \mathbf{c} + \mathcal{E} : \boldsymbol{\sigma} \, d\mathbf{x} + \int_{\partial P_t} \boldsymbol{\mu}^T \mathbf{n} \, da \\ &= \int_{P_t} \mathbf{c} + \mathcal{E} : \boldsymbol{\sigma} + \operatorname{div}(\boldsymbol{\mu}^T) \, d\mathbf{x} \, . \end{aligned} \tag{104}$$

From the localization theorem,

$$0 = \mathbf{c} + \mathcal{E} : \boldsymbol{\sigma} + \operatorname{div}(\boldsymbol{\mu}^T) \tag{105}$$

but since $\mathbf{c} + \operatorname{div}(\boldsymbol{\mu}^T)$ is not necessarily zero, it is not possible to say that $\mathcal{E} : \boldsymbol{\sigma}$ is zero to assert that Cauchy's stress tensor is symmetric.

Institute for Mechanical Systems
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3 Homework

3.1 Material time derivative of integral quantaties

Given a spatial vector field $\mathbf{u}(\mathbf{x}, t)$, show that,

$$\frac{D}{Dt}\int_S \mathbf{u}\, d\mathbf{x} = \int_S \dot{\mathbf{u}} + \mathbf{u}\text{div}\mathbf{v}\, d\mathbf{x} \tag{106}$$

$$= \int_S \mathbf{u}' + \text{div}(\mathbf{u}\otimes\mathbf{v})\, d\mathbf{x} \tag{107}$$

$$= \int_{\partial S} \mathbf{u}(\mathbf{v}\cdot\mathbf{n})\, da + \int_S \mathbf{u}'\, d\mathbf{x} \tag{108}$$

$$\frac{D}{Dt}\int_S \rho\mathbf{u}\, d\mathbf{x} = \int_S \rho\dot{\mathbf{u}}\, d\mathbf{x}\,. \tag{109}$$

Here $\mathcal{S}$ is the spatial configuration of the body $\mathcal{B}$, $\mathbf{v}$ is the spatial velocity, and the $'$ denotes the spatial time derivative.

3.2 Cauchy's theorem

Prove the sufficiency of Cauchy's theorem. Given a continuous traction field $\mathbf{t}(\mathbf{x}, \mathbf{n})$ and body force field (per unit mass) $\mathbf{b}(\mathbf{x})$, show that, the existence of a tensor $\boldsymbol{\sigma}(\mathbf{x})$ such that,

1. $\mathbf{t}(\mathbf{x}, \mathbf{n}) = \boldsymbol{\sigma}^T(\mathbf{x})\mathbf{n}$,
2. $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$,
3. $\text{div}(\boldsymbol{\sigma}^T) + \rho\mathbf{b} = \rho\mathbf{a}$.

implies,

1. Global linear momentum balance,
2. Global angular momentum balance.

3.3 Average stress

Problem:

In the case of a static problem, i.e. $\mathbf{a} = 0$, the balance of equilibrium reduces to,

$$\text{div}(\boldsymbol{\sigma}) + \mathbf{b} \quad = \quad 0\,. \tag{110}$$

Here, the symmetry of $\boldsymbol{\sigma}$ has already been exploited. In such a case the average stress $\overline{\boldsymbol{\sigma}}$ in a body P_t,

$$\overline{\boldsymbol{\sigma}} \quad = \quad \frac{1}{\text{vol}(P_t)} \int_{P_t} \boldsymbol{\sigma}\, d\mathbf{x} \tag{111}$$

can be described in terms of the body force $\mathbf{b}$ and surface traction $\mathbf{t}$,

$$\overline{\boldsymbol{\sigma}} \quad = \quad \frac{1}{2V} \int_{P_t} \mathbf{b} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{b}\, d\mathbf{x} + \frac{1}{2V} \int_{\partial P_t} \mathbf{t} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{t}\, da \tag{112}$$

where $V = \text{vol}(P_t)$. This relation holds irrespective of the material inhomogeneities.

Hint: Try using the following first couple of steps,

$$\begin{aligned} \overline{\boldsymbol{\sigma}}_{ij} \quad &= \quad \frac{1}{\text{vol}(P_t)} \int_{P_t} \sigma_{ij}\, d\mathbf{x} \\ &= \quad \frac{1}{V} \int_{P_t} \sigma_{ik} \delta_{kj}\, d\mathbf{x} \\ &= \quad \frac{1}{V} \int_{P_t} \sigma_{ik} x_{j,k}\, d\mathbf{x}\,. \end{aligned}$$

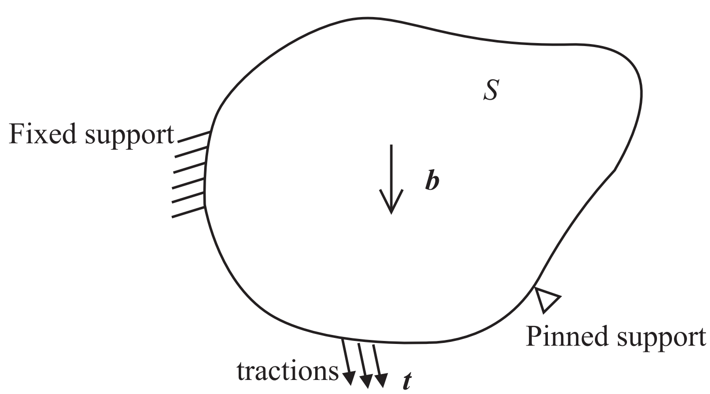

Figure 4: Configuration of the body under traction and body forces