ETH Zurich | Institute for Mechanical Systems
Department of Mechanical and Process Engineering | Center of Mechanics
Winter 06/07
Nonlinear Continuum Mechanics
Exercise 5 | Prof. Dr. Sanjay Govindjee

1 Useful Definitions or Concepts

1.1 Push-forward and pull-backs

Push-forwards and pull-backs are ways to transform an object specifed in the material configuration to one defined in the spatial configuration and vice versa. Such a transformation allows one to work in either the material or spatial configuration, depending on the situation. Here we will introdcue push-forwards and pull-backs for the following cases.

- vectors
- co-vectors(linear functions which take vectors as arguments and return real numbers)
- 2nd-order tensors which take either vectors or co-vectors as arguments and return real numbers

Given a mapping φ from the material configuration $\mathcal{B}$ to the spatial configuration $\mathcal{S}$, the push-forward is denoted,

$$\varphi_*(\ \cdot\ ) \tag{1}$$

and the pull-back is denoted,

$$\varphi^*(\ \cdot\ )\ . \tag{2}$$

1.1.1 Vectors

The push-forward and pull-back of vectors is defined as follows,

$$\varphi_*(\mathbf{V}) = \mathbf{F}\mathbf{V} \quad (\mathbf{V} \in T_{\mathbf{X}}\mathcal{B}) \tag{3}$$
$$\varphi^*(\mathbf{v}) = \mathbf{F}^{-1}\mathbf{v} \quad (\mathbf{v} \in T_{\varphi(\mathbf{X})}\mathcal{S})\ . \tag{4}$$

1.1.2 Co-vectors

Co-vectors are linear functions which take in vectors as arguments and return real numbers. They can be defined for both the material and spatial configuration. The space of co-vectors for the material configuration is denoted,

$$T^*_{\mathbf{X}}\mathcal{B} \tag{5}$$

and for the spatial configuration,

$$T^*_{\varphi(\mathbf{X})}\mathcal{S}\ . \tag{6}$$

Given a co-vector in the material configuration $\mathbf{W} \in T^*_{\mathbf{X}}\mathcal{B}$, its operation on vectors $\mathbf{V} \in T_{\mathbf{X}}\mathcal{B}$ is defined as,

$$\mathbf{W}(\mathbf{V}) := \mathbf{W} \cdot \mathbf{V} \in \mathbb{R}\ . \tag{7}$$

For co-vectors in the spatial configuration $\mathbf{w} \in T^*_{\varphi(\mathbf{X})}\mathcal{S}$, their operation on vector $\mathbf{v} \in T_{\varphi(\mathbf{X})}\mathcal{S}$ is defined as,

$$\mathbf{w}(\mathbf{v}) := \mathbf{w} \cdot \mathbf{v} \in \mathbb{R}\ . \tag{8}$$

The push-forward and pull-back of co-vectors is defined as,

$$\varphi_*(\mathbf{W}) = \mathbf{F}^{-T}\mathbf{W} \quad (\mathbf{W} \in T^*_{\mathbf{X}}\mathcal{B}) \tag{9}$$
$$\varphi^*(\mathbf{w}) = \mathbf{F}^{T}\mathbf{w} \quad (\mathbf{w} \in T^*_{\varphi(\mathbf{X})}\mathcal{S})\ . \tag{10}$$

1.1.3 2nd-order tensors

A tensor can be defined to takes either vectors or co-vectors as arguments, and can be defined either in the material or spatial configuration. This gives 4 possibilities for the form of the 2nd-order tensors.

1. $\boldsymbol{\sigma} : T_{\mathbf{X}}\mathcal{B} \times T_{\mathbf{X}}\mathcal{B} \longmapsto \mathbb{R}$
2. $\boldsymbol{\sigma} : T_{\boldsymbol{\varphi}(\mathbf{X})}\mathcal{S} \times T_{\boldsymbol{\varphi}(\mathbf{X})}\mathcal{S} \longmapsto \mathbb{R}$
3. $\boldsymbol{\sigma} : T^*_{\mathbf{X}}\mathcal{B} \times T^*_{\mathbf{X}}\mathcal{B} \longmapsto \mathbb{R}$
4. $\boldsymbol{\sigma} : T^*_{\boldsymbol{\varphi}(\mathbf{X})}\mathcal{S} \times T^*_{\boldsymbol{\varphi}(\mathbf{X})}\mathcal{S} \longmapsto \mathbb{R}$

1,2 are the push-forward and pull-back of tensors acting on vectors and can be considered one pair.

1. $\boldsymbol{\varphi}_*(\boldsymbol{\sigma}) = \mathbf{F}^{-T}\boldsymbol{\sigma}\mathbf{F}^{-1}$
2. $\boldsymbol{\varphi}^*(\boldsymbol{\sigma}) = \mathbf{F}^{T}\boldsymbol{\sigma}\mathbf{F}$

3,4 are the push-forward and pull-back of tensors acting on co-vectors and can be considered another pair.

3. $\boldsymbol{\varphi}_*(\boldsymbol{\sigma}) = \mathbf{F}\boldsymbol{\sigma}\mathbf{F}^{T}$
4. $\boldsymbol{\varphi}^*(\boldsymbol{\sigma}) = \mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$

1.2 The Lie derivative

The Lie derivative of a spatial tensor is defined as follows.

$$\mathcal{L}_{\mathbf{v}}(\ \cdot\ ) \;=\; \boldsymbol{\varphi}_*\left(\frac{D}{Dt}\left(\boldsymbol{\varphi}^*(\ \cdot\ )\right)\right) \tag{11}$$

1.3 Spatial velocity gradient and rates

Given the spatial velocity $\mathbf{v}$, the spatial velocity gradient $\mathbf{l}$ is defined as,

$$\begin{aligned} \mathbf{l} &= \frac{\partial \mathbf{v}}{\partial \mathbf{x}} && (12) \\ &= \frac{\partial \mathbf{v}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{x}} \\ &= \frac{\partial \dot{\varphi}}{\partial \mathbf{X}}\frac{\partial \mathbf{X}}{\partial \mathbf{x}} \\ &= \dot{\mathbf{F}}\mathbf{F}^{-1}\,. && (13) \end{aligned}$$

The spatial velocity gradient gives information concerning the instantaneous change of the motion. $\mathbf{l}$ can be decomposed into its symmetric part $\mathbf{d}$ (rate of deformation tensor) and skew part $\mathbf{w}$ (spin tensor).

$$\mathbf{l} \;=\; \mathbf{d} + \mathbf{w} \tag{14}$$

As the name suggests, the rate of deformation tensor contains information concerning the rate at which the material locally deforms, and the spin tensor contains information concerning the rate of rotation.

- The eigenvalues of $\mathbf{d}$ denote the rate of stretch along the direction of the eigenvectors of $\mathbf{d}$.
- The axial vector $\mathbf{w}$ denotes the axis and rate of the rotation.

The action of $\mathbf{l}$ can be interpreted by as a pure triaxial stretch in the direction of the eigenvectors of $\mathbf{d}$ and a rigid rotation around the axial vector of $\mathbf{w}$.

2 Homework

2.1 Quantaties related to rates

Problem:

Given the spin tensor $\mathbf{w}$, show that its axial vector $\boldsymbol{\omega}$ defined as,

$$\mathbf{w}\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a} \tag{15}$$

can be represented as,

$$\boldsymbol{\omega} = \frac{1}{2}\text{curl}\mathbf{v}\ . \tag{16}$$

This shows that the axial vector of the spin tensor is half of the vorticity.

Solution:

Let $\mathbf{a}$ be and arbitrary vector. We have,

$$\begin{aligned}
\mathbf{w}\mathbf{a} &= \frac{1}{2}\left(\mathbf{l}+\mathbf{l}^T\right)\mathbf{a} \\
&= \frac{1}{2}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T\right)\mathbf{a} \\
w_{ij}a_j &= \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)a_j
\end{aligned} \tag{17}$$

and,

$$\begin{aligned}
\boldsymbol{\omega} \times \mathbf{a} &= \frac{1}{2}\text{curl}\mathbf{v} \times \mathbf{a} \\
&= \frac{1}{2}\left(\nabla \times \mathbf{v}\right) \times \mathbf{a} \\
&= \frac{1}{2}\left(v_{j,i}\varepsilon_{ijk}\mathbf{e}_k\right) \times a_l\mathbf{e}_l \\
&= \frac{1}{2}v_{j,i}a_l\varepsilon_{ijk}\varepsilon_{klm}\mathbf{e}_m \\
&= \frac{1}{2}v_{j,i}a_l\varepsilon_{ijk}\varepsilon_{lmk}\mathbf{e}_m \\
&= \frac{1}{2}v_{j,i}a_l\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\mathbf{e}_m \\
&= \frac{1}{2}\left(v_{m,l}a_l - v_{l,m}a_l\right)\mathbf{e}_m \\
&= \frac{1}{2}\left(v_{m,l} - v_{l,m}\right)a_l\mathbf{e}_m \\
\left(\boldsymbol{\omega} \times \mathbf{a}\right)_m &= \frac{1}{2}\left(v_{m,l} - v_{l,m}\right)a_l\ .
\end{aligned} \tag{18}$$

This shows that the expressions are equivalent.

Problem:

Show that,

$$\begin{aligned} \dot{da} &= \left(\text{tr}\mathbf{l} - \mathbf{n}\cdot\mathbf{l}\mathbf{n}\right) da & (19) \\ \dot{\mathbf{n}} &= \left(\mathbf{n}\cdot\mathbf{l}\mathbf{n}\right)\mathbf{n} - \mathbf{l}^T\mathbf{n} & (20) \end{aligned}$$

where da is a spatial area element and $\mathbf{n}$ is the unit normal to this surface element. Recall that we have the relation from Nansons's formula relating the material and spatial area elements and normals.

$$\mathbf{n}da = J\mathbf{F}^{-T}\mathbf{N}dA\,. \tag{21}$$

Hint: Express da in terms of quantaties $J, \mathbf{N}, \mathbf{C}, \mathbf{N}, dA$ and take a material time derivative. Recall that the material time derivatives of $\mathbf{N}, dA$ are zero. Do the calculation in symbolic notation. It can become quite messy if you try this in index notation.

Solution:

Using Nanson's formula,

$$\begin{aligned} \mathbf{n}da\cdot\mathbf{n}da &= \left(J\mathbf{F}^{-T}\mathbf{N}dA\right)\cdot\left(J\mathbf{F}^{-T}\mathbf{N}dA\right) \\ da^2 &= J^2\mathbf{F}^{-T}\mathbf{N}\cdot\mathbf{F}^{-T}\mathbf{N}dA^2 \\ &= J^2\mathbf{N}\cdot\mathbf{F}^{-1}\mathbf{F}^{-T}\mathbf{N}dA^2\,. & (22) \end{aligned}$$

Since,

$$\begin{aligned} \overline{F^{-1}}^{\,\cdot} &= -\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1} & (23) \\ \overline{F^{-1}F^{-T}}^{\,\cdot} &= -\mathbf{F}^{-1}\mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{F}^{-T} - \mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{F}^{-T} \\ &= -\mathbf{F}^{-1}\left(\mathbf{l}^T+\mathbf{l}\right)\mathbf{F}^{-T} \\ &= -2\mathbf{F}^{-1}\mathbf{d}\mathbf{F}^{-T} & (24) \\ \dot{J} &= \frac{\partial J}{\partial \mathbf{F}} : \dot{\mathbf{F}} \\ &= J\mathbf{F}^{-T} : \dot{\mathbf{F}} \\ &= J\mathbf{1} : \dot{\mathbf{F}}\mathbf{F}^{-1} \\ &= J\mathbf{1} : \mathbf{l} \\ &= J\text{tr}\mathbf{l} & (25) \end{aligned}$$

taking a material time derivative of the expression for da^2 yields,

$$\begin{aligned} 2da\dot{da} &= 2J\dot{J}\mathbf{N}\cdot\mathbf{F}^{-1}\mathbf{F}^{-T}\mathbf{N}dA^2 + J^2\mathbf{N}\cdot\overline{(\mathbf{F}^{-1}\mathbf{F}^{-T})}^{\,\cdot}\mathbf{N}dA^2 \\ &= 2J^2\text{tr}\mathbf{l}\;\mathbf{N}\cdot\mathbf{F}^{-1}\mathbf{F}^{-T}\mathbf{N}dA^2 - 2J^2\mathbf{N}\cdot\mathbf{F}^{-1}\mathbf{d}\mathbf{F}^{-T}\mathbf{N}dA^2 \\ &= 2\text{tr}\;\mathbf{l}\,da^2 - 2J^2\mathbf{F}^{-T}\mathbf{N}\cdot\mathbf{d}\mathbf{F}^{-T}\mathbf{N}dA^2 \\ &= 2\text{tr}\mathbf{l}\;da^2 - 2\left(J\mathbf{F}^{-T}\mathbf{N}dA\right)\cdot\mathbf{d}\left(J\mathbf{F}^{-T}\mathbf{N}dA\right) \\ &= 2\text{tr}\mathbf{l}\,da^2 - 2\mathbf{n}da\cdot\mathbf{d}\mathbf{n}da \\ \dot{da} &= \text{tr}\mathbf{l}da - \mathbf{n}\cdot\mathbf{d}\mathbf{n}da \\ &= \left(\text{tr}\mathbf{l} - \mathbf{n}\cdot\mathbf{d}\mathbf{n}\right)da \\ &= \left(\text{tr}\mathbf{l} - \mathbf{n}\cdot\mathbf{l}\mathbf{n}\right)da\,. & (26) \end{aligned}$$

In the equation $\mathbf{n} \cdot \mathbf{l}\mathbf{n} = \mathbf{n} \cdot \mathbf{d}\mathbf{n}$, since $\mathbf{w}$ is skew.

Next we take the material time derivative of $\mathbf{n}da$.

$$\begin{aligned}
\dot{\overline{\mathbf{n}da}} &= \dot{\overline{J\mathbf{F}^{-T}\mathbf{N}dA}} \\
\dot{\mathbf{n}}da + \mathbf{n}\dot{da} &= \dot{J}\mathbf{F}^{-T}\mathbf{N}dA + J\dot{\overline{\mathbf{F}^{-T}}}\mathbf{N}dA \\
&= J\text{tr}\mathbf{l}\ \mathbf{F}^{-T}\mathbf{N}dA - J\mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{F}^{-T}\mathbf{N}dA \\
&= \text{tr}\mathbf{l}\ \mathbf{n}da - J\mathbf{l}^T\mathbf{F}^{-T}\mathbf{N}dA \\
&= \text{tr}\mathbf{l}\ \mathbf{n}da - \mathbf{l}^T\mathbf{n}da \\
\dot{\mathbf{n}}da + \mathbf{n}\left(\text{tr}\mathbf{l} - \mathbf{n}\cdot\mathbf{l}\mathbf{n}\right)da &= \text{tr}\mathbf{l}\ \mathbf{n}da - \mathbf{l}^T\mathbf{n}da \\
\dot{\mathbf{n}}da &= \mathbf{n}\cdot\mathbf{l}\mathbf{n}da - \mathbf{l}^T\mathbf{n}da \\
\dot{\mathbf{n}} &= \mathbf{n}\cdot\mathbf{l}\mathbf{n} - \mathbf{l}^T\mathbf{n}\,. \qquad (27)
\end{aligned}$$

Remark:

In this problem it must be pointed out that the normal vector $\mathbf{n}$ is in fact not a vector but has the properties of a co-vector. Thus it does not transform the in the same manner as a vector, and the relationship,

$$\dot{\mathbf{a}} = \mathbf{l}\mathbf{a} \qquad (28)$$

does not hold. The normal vector is and co-vector and given a normal vector $\mathbf{B}$ in the reference configuration, it is mapped to $\mathbf{b}$ in the spatial configuration as,

$$\mathbf{b} = \mathbf{F}^{-T}\mathbf{B}\,. \qquad (29)$$

This can be seen by the following example of simple shear in 2D.

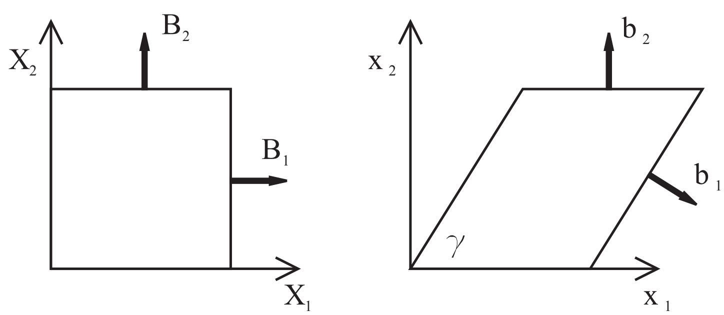

Figure 1: Simple shear

The motion and deformation gradient are given as,

$$
\begin{aligned}
\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} &= \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \\
\mathbf{F} &= \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \\
\mathbf{F}^{-T} &= \begin{bmatrix} 1 & 0 \\ -\gamma & 1 \end{bmatrix} .
\end{aligned}
$$

Denote the normal to the right edge of the square as $\mathbf{B}_1$ and the top edge as $\mathbf{B}_2$.

$$
\begin{aligned}
\mathbf{B}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\mathbf{B}_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\mathbf{F}\mathbf{B}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\mathbf{F}\mathbf{B}_2 &= \begin{bmatrix} \gamma \\ 1 \end{bmatrix} \\
\mathbf{F}^{-T}\mathbf{B}_1 &= \begin{bmatrix} 1 \\ -\gamma \end{bmatrix} \\
\mathbf{F}^{-T}\mathbf{B}_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

Clearly the objects mapped with $\mathbf{F}$ do not represent the normal vectors in the spatial configuration. On the contrary the $\mathbf{F}^{-T}$ gives the correct result.

Another interpretation of the surface normal as a co-vector follows from the following. Let $\mathbf{A}$ be a tangent vector, and $\mathbf{B}$ be the normal to this. Then, $\mathbf{A} \cdot \mathbf{B} = 0$. Let $\mathbf{a}, \mathbf{b}$ be the mapped versions in the spatial configuration. $\mathbf{a} \cdot \mathbf{b} = 0$ is desired. For this,

$$
\begin{aligned}
0 &= \mathbf{a} \cdot \mathbf{b} \\
&= \mathbf{F}\mathbf{A} \cdot \mathbf{b} \\
&= \mathbf{A} \cdot \mathbf{F}^T\mathbf{b}
\end{aligned} \tag{30}
$$

and,

$$
\begin{aligned}
\mathbf{B} &= \mathbf{F}^T\mathbf{b} \\
\mathbf{b} &= \mathbf{F}^{-T}\mathbf{B} .
\end{aligned} \tag{31}
$$

Thus it is clear that normal vectors map as co-vectors with $\mathbf{F}^{-T}$. The transformation for objects that are tangent to lines and objects that are normal to lines are different.

Let us compute the relation between the time rate change of a co-vector and the co-vector.

$$
\begin{aligned}
\dot{\mathbf{b}} &= \frac{D}{Dt}(\mathbf{F}^{-T}\mathbf{B}) \\
&= \overline{\mathbf{F}^{-T}}^{\cdot}\mathbf{B} \\
&= -\mathbf{F}^{-T}\dot{\mathbf{F}}^T\mathbf{F}^{-T}\mathbf{B} \\
&= -\mathbf{l}^T\mathbf{b}
\end{aligned} \tag{32}
$$

Thus the mapping is given with $-\mathbf{l}^T$ and not $\mathbf{l}$.

The time rate of change for vectors and co-vectors that are constrained to unity are slightly different from the case for arbitrary vectors. Recall that for a unit vector $\mathbf{m}$ the mapping is given not by,

$$\dot{\mathbf{m}} = \mathbf{l}\mathbf{m} \tag{33}$$

but by

$$\dot{\mathbf{m}} = (\mathbf{l} - \mathbf{m}\cdot\mathbf{l}\mathbf{m}\mathbf{1})\mathbf{m}\,. \tag{34}$$

This can be interpreted as a mapping by $\mathbf{l}$ and then a modification made to retain orthogonality with $\mathbf{m}$ so that $\mathbf{m}\cdot\dot{\mathbf{m}} = 0$.

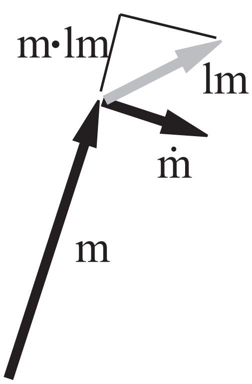

Figure 2: Time rate of change of vector

The time rate of change for co-vectors can be obtained similarly to the case of vectors. Let,

$$\mathbf{b} = \mathbf{n}\,ds \tag{35}$$

where $\mathbf{n}$ is a unit vector pointing in the direction of $\mathbf{b}$ and ds is its length. For the vector $\mathbf{b}$,

$$\begin{aligned} \dot{\mathbf{b}} &= -\mathbf{l}^T\mathbf{b} \\ \overline{\mathbf{n}\,ds}^{\boldsymbol{\cdot}} &= -\mathbf{l}^T\mathbf{n}\,ds \\ \dot{\mathbf{n}}\,ds + \mathbf{n}\dot{ds} &= -\mathbf{l}^T\mathbf{n}\,ds\,. \end{aligned} \tag{36}$$

Taking the inner the product with $\mathbf{n}$ and using the relation $\dot{\mathbf{n}}\cdot\mathbf{n} = 0$, one obtains,

$$\mathbf{n}\dot{ds} = -\mathbf{n}\cdot\mathbf{l}^T\mathbf{n}\,ds. \tag{37}$$

Reinserting this into eqn. (36), one obtains

$$\begin{aligned} \dot{\mathbf{n}}\,ds + \mathbf{n}(-\mathbf{n}\cdot\mathbf{l}^T\mathbf{n})\,ds &= -\mathbf{l}^T\mathbf{n}\,ds \\ \dot{\mathbf{n}} &= -\mathbf{l}^T\mathbf{n} - \mathbf{n}(-\mathbf{n}\cdot\mathbf{l}^T\mathbf{n}) \\ \dot{\mathbf{n}} &= \left(-\mathbf{l}^T - \mathbf{n}\cdot(-\mathbf{l}^T)\mathbf{n}\right)\mathbf{1})\mathbf{n}\,. \end{aligned} \tag{38}$$

This equation resembles closely the case for vectors but with the operator $\mathbf{l}$ replaced by $-\mathbf{l}^T$.

Institute for Mechanical Systems
Center of Mechanics

Prof. Dr. Sanjay Govindjee

Problem:

A deformation which is volume preserving is called isochoric or incompressible. The following are all criterion for isochoric motion.

$$J = 1 \tag{39}$$
$$\dot{J} = 0 \tag{40}$$
$$\mathbf{F}^{-T} : \dot{\mathbf{F}} = 0 \tag{41}$$
$$\text{tr}\mathbf{l} = 0 \tag{42}$$
$$\text{tr}\mathbf{d} = 0 \tag{43}$$
$$\text{div}\mathbf{v} = 0 \tag{44}$$

Explain why these expressions are equivalent. In other words give the reason why one criterion implies the other word.

Then show that the time rate of change of a volume element dv in the spatial configuration is given by the following expression.

$$\dot{dv} = \text{tr}(\mathbf{l})dv \tag{45}$$
$$= \text{tr}(\mathbf{d})dv \tag{46}$$

Hint: State the relation between the spatial volume element dv and the material volume element dV. Then take a material time derivative of this expression.

Solution:

Volume is preserved in the motion if,

$$J = \det\mathbf{F} = 1\ . \tag{47}$$

Alternatively,

$$\begin{aligned} 0 &= \dot{J} \\ &= J\mathbf{F}^{-T} : \dot{\mathbf{F}} \\ &= J\mathbf{1} : \dot{\mathbf{F}}\mathbf{F}^{-T} \\ &= J\text{tr}\mathbf{l} \\ &= J\text{tr}\mathbf{d} \\ &= J\text{div}\mathbf{v}\ . \end{aligned} \tag{48}$$

Thus the following criterion are equivalent,

$$J = 1 \tag{49}$$
$$\dot{J} = 0 \tag{50}$$
$$\mathbf{F}^{-T} : \dot{\mathbf{F}} = 0 \tag{51}$$
$$\text{tr}\mathbf{l} = 0 \tag{52}$$
$$\text{tr}\mathbf{d} = 0 \tag{53}$$
$$\text{div}\mathbf{v} = 0 \tag{54}$$

By taking a derivative of the relation between the spatial and material volume elements.

$$\begin{aligned} \dot{dv} &= \overline{J dV}^{\cdot} \\ &= \dot{J} dV \\ &= J\mathbf{F}^{-T} : \dot{\mathbf{F}} dV \\ &= \mathbf{F}^{-T} : \dot{\mathbf{F}} dv \\ &= \mathbf{1} : \dot{\mathbf{F}}\mathbf{F}^{-1} dv \\ &= \text{tr}(\mathbf{l}) dv & (55) \\ &= \text{tr}(\mathbf{d}) dv & (56) \end{aligned}$$

The last line is true since $\text{tr}\mathbf{w}$, the trace of the spin tensor, is zero. Thus we see that the relative change in volume is given by the trace of the rate of deformation tensor.