ETH Zurich
Department of Mechanical and Process Engineering
Winter 06/07
Nonlinear Continuum Mechanics
Exercise 4

Institute for Mechanical Systems
Center of Mechanics

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1 Useful Definitions or Concepts

1.1 Material and Spatial Descriptions

Functions can have arguments which depend on material description $\mathbf{X}$ and spatial description $\mathbf{x}$. They can be transformed from one to the other through the mapping,

$$\mathbf{x} = \varphi(\mathbf{X}, t) \quad (1)$$
$$\mathbf{X} = \varphi^{-1}(\mathbf{x}, t) \quad (2)$$

between the two configurations. Using this relation, any function can be written in both descriptions. Let $F(\mathbf{X})$ be a scalar-valued function in the material description. If we define the name of the spatial version as $f(\mathbf{x})$, then the relation between these becomes,

$$f(\mathbf{x}) = F(\varphi^{-1}(\mathbf{x}))\ . \quad (3)$$

The velocity and acceleration are defined as follows,

$$\mathbf{V}(\mathbf{X}, t) = \left(\frac{\partial \varphi(\mathbf{X}, t)}{\partial t}\right)_{\mathbf{X}} \quad (4)$$
$$\mathbf{A}(\mathbf{X}, t) = \left(\frac{\partial \mathbf{V}(\mathbf{X}, t)}{\partial t}\right)_{\mathbf{X}} = \left(\frac{\partial^2 \varphi(\mathbf{X}, t)}{\partial^2 t}\right)_{\mathbf{X}} \quad (5)$$

by taking time derivatives of the mapping. These functions can also be expressed in terms of the spatial description by inserting the inverse mapping,

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{V}\left(\varphi^{-1}(\mathbf{x}, t), t\right) \quad (6)$$
$$\mathbf{a}(\mathbf{x}, t) = \mathbf{A}\left(\varphi^{-1}(\mathbf{x}, t), t\right)\ . \quad (7)$$

What one would like to point out is that,

$$\mathbf{a}(\mathbf{x}, t) \neq \left(\frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t}\right)_{\mathbf{x}}\ . \quad (8)$$

1.2 Material and Spatial Time derivatives

When taking the time derivative of a function F, we can take a material time derivative or a spatial time derivative.

- Material: Put a dot over the function, or use Upper Case "D".

$$\left(\frac{\partial F}{\partial t}\right)_{\mathbf{X}} = \dot{F} = \frac{DF}{Dt} \quad (9)$$

- Spatial: Put a $'$ by the function, or just use "∂".

$$\left(\frac{\partial F}{\partial t}\right)_{\mathbf{x}} = F' = \frac{\partial F}{\partial t} \quad (10)$$

The case where we take a material time derivative of a function expressed in spatial description is slightly more involved.

$$\begin{aligned}
\left(\frac{\partial F(\mathbf{x},t)}{\partial t}\right)_{\mathbf{X}} &= \left(\frac{\partial F(\varphi(\mathbf{X},t),t)}{\partial t}\right)_{\mathbf{X}} \\
&= \left(\frac{\partial F(\varphi(\mathbf{X},t),t)}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial F(\varphi(\mathbf{X},t),t)}{\partial \mathbf{x}}\right)_{t} \left(\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}\right)_{\mathbf{X}} \\
&= \left(\frac{\partial F(\varphi(\mathbf{X},t),t)}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial F(\varphi(\mathbf{X},t),t)}{\partial \mathbf{x}}\right)_{t} \cdot \mathbf{V}(\mathbf{X},t) \\
&= \left(\frac{\partial F(\mathbf{x},t)}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial F(\mathbf{x},t)}{\partial \mathbf{x}}\right)_{t} \cdot \mathbf{v}(\mathbf{x},t) \qquad (11)
\end{aligned}$$

1.2.1 The inverse of the deformation gradient

Let a motion $\boldsymbol{\varphi}$ be given that maps points from the material configuration $\mathcal{B}$ to the spatial configuration $\mathcal{S}$.

$$\boldsymbol{\varphi} : \mathcal{B} \rightarrow \mathcal{S} \qquad (12)$$

The deformation gradient for this motion is,

$$\mathbf{F} = \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{X}} \, . \qquad (13)$$

Since we assume that the motion is one-one and onto,

- One-one: Two points from $\mathcal{B}$ do not go to the same point in $\mathcal{S}$. This physically means that the material does not overlap itself.
- Onto: Everypoint in $\mathcal{S}$ comes from some point in $\mathcal{B}$. This physically means that points in $\mathcal{S}$ are not magically generated from nowhere and always must come from some point in $\mathcal{B}$.

we can define the inverse map $\boldsymbol{\varphi}^{-1}$,

$$\mathbf{X} = \boldsymbol{\varphi}^{-1}(\mathbf{x}) \, . \qquad (14)$$

By differentiating this expression with respect to $\mathbf{x}$ we have,

$$\frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \frac{\partial \boldsymbol{\varphi}^{-1}}{\partial \mathbf{x}} \, . \qquad (15)$$

From the relation between the mapping and inverse mapping, we can write the two expressions,

$$\begin{aligned}
\boldsymbol{\varphi}^{-1}(\boldsymbol{\varphi}(\mathbf{X})) &= \mathbf{X} \qquad (16) \\
\boldsymbol{\varphi}(\boldsymbol{\varphi}^{-1}(\mathbf{x})) &= \mathbf{x} \, . \qquad (17)
\end{aligned}$$

Let us differentiate the first expression by $\mathbf{X}$. By chain rule,

$$\begin{aligned}
\frac{\partial \boldsymbol{\varphi}^{-1}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\boldsymbol{\varphi} \mathbf{X}} &= \mathbf{1} \\
\frac{\partial \boldsymbol{\varphi}^{-1}}{\partial \mathbf{x}} \mathbf{F} &= \\
\frac{\partial \mathbf{X}}{\partial \mathbf{x}} \mathbf{F} &= \qquad . \qquad (18)
\end{aligned}$$

If we diffentiate the second expression by $\mathbf{x}$ we have,

$$\begin{aligned} \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{X}} \frac{\partial \mathbf{X}}{\partial \mathbf{x}} &= \mathbf{1} \\ \mathbf{F} \frac{\partial \mathbf{X}}{\partial \mathbf{x}} &= \qquad . \end{aligned} \tag{19}$$

These two relations imply that,

$$\mathbf{F}^{-1} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}} \tag{20}$$

$$= \frac{\partial \boldsymbol{\varphi}^{-1}}{\partial \mathbf{x}} \, . \tag{21}$$

1.2.2 Interpretation of deformation gradient

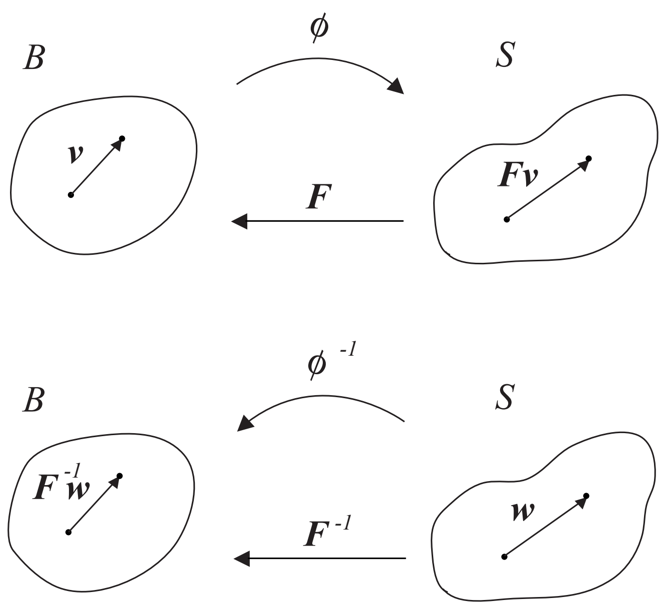

Figure 1: The mapping between tangent spaces

Let a motion φ be given that maps points from the material configuration $\mathcal{B}$ to the spatial configuration $\mathcal{S}$.

$$\varphi : \mathcal{B} \rightarrow \mathcal{S} \tag{22}$$

The important interpretation of the deformation gradient,

$$\mathbf{F} = \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{X}} \tag{23}$$

is that it maps vectors from the material configuration to the spatial configuration. To say this a little more precisely we introduce the notation of a ”Tangent Space”.

$$T_{\mathbf{X}}\mathcal{B}: \quad \text{vectors eminating from the point x in the space B} \tag{24}$$

With this the deformation gradient can be expressed as,

$$\mathbf{F}(\mathbf{X}): \quad T_{\mathbf{X}}\mathcal{B} \rightarrow \quad T_{\boldsymbol{\varphi}(\mathbf{X})}\mathcal{S} \tag{25}$$

such that,

$$\mathbf{F}: \mathbf{v} \longmapsto \mathbf{F}\mathbf{v} \quad . \tag{26}$$

The inverse of the deformation gradient $\mathbf{F}^{-1}$ does the opposite,

$$\mathbf{F}^{-1}(\mathbf{x}): \quad T_{\mathbf{x}}\mathcal{S} \rightarrow \quad T_{\boldsymbol{\varphi}^{-1}(\mathbf{x})}\mathcal{B} \tag{27}$$

such that,

$$\mathbf{F}^{-1}: \mathbf{w} \longmapsto \mathbf{F}^{-1}\mathbf{w} \quad . \tag{28}$$

1.3 Applications of concepts or definitions

Problem:

Given the rigid body motion,

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\varphi}(\mathbf{X}, t) \\ &= \mathbf{Q}(t)\mathbf{X} + c(t) \end{aligned} \tag{29}$$

compute the material velocity $\mathbf{V}$, material acceleration $\mathbf{A}$, spatial velocity $\mathbf{v}$, and spatial acceleration $\mathbf{a}$. Confirm the relation between the material time derivative and the spatial time derivative of the spatial velocity.

Solution:

The material velocity and acceleration can be obtained from sequentially taking the material time derivative of the motion $\boldsymbol{\varphi}$.

$$\begin{aligned} \mathbf{V}(\mathbf{X}, t) &= \left(\frac{\partial \boldsymbol{\varphi}(\mathbf{X}, t)}{\partial t}\right)_{\mathbf{X}} \\ &= \dot{\mathbf{Q}}(t)\mathbf{X} + \dot{c}(t) \end{aligned} \tag{30}$$

$$\begin{aligned} \mathbf{A}(\mathbf{X}, t) &= \left(\frac{\partial \mathbf{V}(\mathbf{X}, t)}{\partial t}\right)_{\mathbf{X}} \\ &= \left(\frac{\partial \dot{\boldsymbol{\varphi}}(\mathbf{X}, t)}{\partial t}\right)_{\mathbf{X}} \\ &= \ddot{\mathbf{Q}}(t)\mathbf{X} + \ddot{c}(t) \end{aligned} \tag{31}$$

Given the motion, the material coordinates $\mathbf{X}$ can be expressed in terms of the spatial coordinates $\mathbf{x}$ by inverting the mapping $\boldsymbol{\varphi}$,

$$\begin{aligned} \mathbf{X}(\mathbf{x}, t) &= \boldsymbol{\varphi}^{-1}(\mathbf{x}, t) \\ &= \mathbf{Q}^{-1}(\mathbf{x} - \mathbf{c}(t)) \\ &= \mathbf{Q}^{T}(\mathbf{x} - \mathbf{c}(t))\ . \end{aligned} \tag{32}$$

Using this relationship, we obtain the desired quantaties,

$$\begin{aligned} \mathbf{v}(\mathbf{x}, t) &= \mathbf{V}(\boldsymbol{\varphi}^{-1}(\mathbf{x}, t), t) \\ &= \dot{\mathbf{Q}}(t)\mathbf{Q}^{T}\left((\mathbf{x} - c(t)) + \dot{c}(t)\right. \\ \mathbf{a}(\mathbf{x}, t) &= \mathbf{A}(\boldsymbol{\varphi}^{-1}(\mathbf{x}, t), t) \\ &= \ddot{\mathbf{Q}}(t)\mathbf{Q}^{T}\left((\mathbf{x} - c(t)) + \ddot{c}(t)\right.\ . \end{aligned} \tag{33}$$

Next we confirm the relation,

$$\begin{aligned} \left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{X}} &= \left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)_{t}\left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{X}} \\ &= \left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)_{t} \cdot \mathbf{v}\ . \end{aligned} \tag{34}$$

From the computations above we have,

$$\begin{aligned}\left(\frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t}\right)_{\mathbf{X}} &= \left(\frac{\partial \mathbf{V}(\mathbf{X},t)}{\partial t}\right)_{\mathbf{X}} \\ &= \mathbf{a}(\mathbf{x},t) \\ &= \ddot{\mathbf{Q}}(t)\mathbf{Q}^T\left(\mathbf{x}-c(t)\right)+\ddot{c}(t)\,. \end{aligned} \tag{35}$$

For the left hand side of the equation,

$$\begin{aligned}\left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{x}}+\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)_{t}\cdot\mathbf{v} &= \left(\frac{\partial\left(\dot{\mathbf{Q}}(t)\mathbf{Q}^T\left(\mathbf{x}-c(t)\right)+\dot{c}(t)\right)}{\partial t}\right)_{\mathbf{x}} \\ &\quad+\left(\frac{\partial\left(\dot{\mathbf{Q}}(t)\mathbf{Q}^T\left(\mathbf{x}-c(t)\right)+\dot{c}(t)\right)}{\partial \mathbf{x}}\right)_{\mathbf{t}}\left(\dot{\mathbf{Q}}(t)\mathbf{Q}^T\left(\mathbf{x}-c(t)\right)+\dot{c}(t)\right) \\ &= \left(\ddot{\mathbf{Q}}(t)\mathbf{Q}^T\left(\mathbf{x}-c(t)\right)+\dot{\mathbf{Q}}(t)\dot{\mathbf{Q}}^T\left(\mathbf{x}-c(t)\right)+\dot{\mathbf{Q}}(t)\mathbf{Q}^T\left((-\dot{c}(t))+\ddot{c}(t)\right)\right. \\ &\quad+\left(\dot{\mathbf{Q}}(t)\mathbf{Q}^T\right)\left(\dot{\mathbf{Q}}(t)\mathbf{Q}^T\left((\mathbf{x}-c(t))+\dot{c}(t)\right)\right) \\ &= \left(\ddot{\mathbf{Q}}(t)\mathbf{Q}^T+\dot{\mathbf{Q}}(t)\dot{\mathbf{Q}}^T\right)\left(\mathbf{x}-c(t)\right)-\dot{\mathbf{Q}}(t)\mathbf{Q}^T\dot{c}(t)+\ddot{c}(t) \\ &\quad+\dot{\mathbf{Q}}(t)\mathbf{Q}^T\dot{\mathbf{Q}}(t)\mathbf{Q}^T\left(\mathbf{x}-c(t)\right)+\dot{\mathbf{Q}}(t)\mathbf{Q}^T\dot{c}(t) \end{aligned} \tag{36}$$

For an orthogonal tensor $\mathbf{Q}(t)$ we have,

$$\mathbf{Q}(t)\mathbf{Q}^T(t)=\mathbf{1} \tag{37}$$

and if we differentiate this relationship with respect to time we obtain,

$$\begin{aligned}\dot{\mathbf{Q}}(t)\mathbf{Q}^T(t)+\mathbf{Q}(t)\dot{\mathbf{Q}}^T(t) &= \mathbf{0} \\ \dot{\mathbf{Q}}(t)\mathbf{Q}^T(t) &= -\mathbf{Q}(t)\dot{\mathbf{Q}}^T(t) \\ \dot{\mathbf{Q}}(t)\mathbf{Q}^T(t) &= -\left(\dot{\mathbf{Q}}(t)\mathbf{Q}^T(t)\right)^T\,. \end{aligned} \tag{38}$$

We see that $\dot{\mathbf{Q}}(t)\mathbf{Q}^T(t)$ is a skew tensor. Inserting this relationship in eqn. (36),

$$\begin{aligned}\left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{x}}+\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)_{t}\cdot\mathbf{v} &= \left(\ddot{\mathbf{Q}}(t)\mathbf{Q}^T+\dot{\mathbf{Q}}(t)\dot{\mathbf{Q}}^T\right)\left(\mathbf{x}-c(t)\right)-\dot{\mathbf{Q}}(t)\mathbf{Q}^T\dot{c}(t)+\ddot{c}(t) \\ &\quad+\dot{\mathbf{Q}}(t)\mathbf{Q}^T\dot{\mathbf{Q}}(t)\mathbf{Q}^T\left(\mathbf{x}-c(t)\right)+\dot{\mathbf{Q}}(t)\mathbf{Q}^T\dot{c}(t) \\ &= \left(\ddot{\mathbf{Q}}(t)\mathbf{Q}^T+\dot{\mathbf{Q}}(t)\dot{\mathbf{Q}}^T\right)\left(\mathbf{x}-c(t)\right)+\ddot{c}(t) \\ &\quad-\dot{\mathbf{Q}}(t)\mathbf{Q}^T\mathbf{Q}\dot{\mathbf{Q}}^T(t)\left(\mathbf{x}-c(t)\right) \\ &= \left(\ddot{\mathbf{Q}}(t)\mathbf{Q}^T+\dot{\mathbf{Q}}(t)\dot{\mathbf{Q}}^T\right)\left(\mathbf{x}-c(t)\right)+\ddot{c}(t) \\ &\quad-\dot{\mathbf{Q}}(t)\dot{\mathbf{Q}}^T(t)\left(\mathbf{x}-c(t)\right) \\ &= \left(\ddot{\mathbf{Q}}(t)\mathbf{Q}^T\right)\left(\mathbf{x}-c(t)\right)+\ddot{c}(t)\,. \end{aligned} \tag{39}$$

Thus the relation between the material and spatial time derivatives have been confirmed.

Problem:

What is the relation between the vector between two points in the material configuration $\mathbf{X}, \mathbf{Y}$, and the vector between the mapped points $\mathbf{x}, \mathbf{y}$ in the spatial configuration. Show that when the distance between $\mathbf{X}$ and $\mathbf{Y}$ is "small", that,

$$\mathbf{y} - \mathbf{x} \quad \approx \quad \mathbf{F}\left(\mathbf{Y} - \mathbf{X}\right) \ . \tag{40}$$

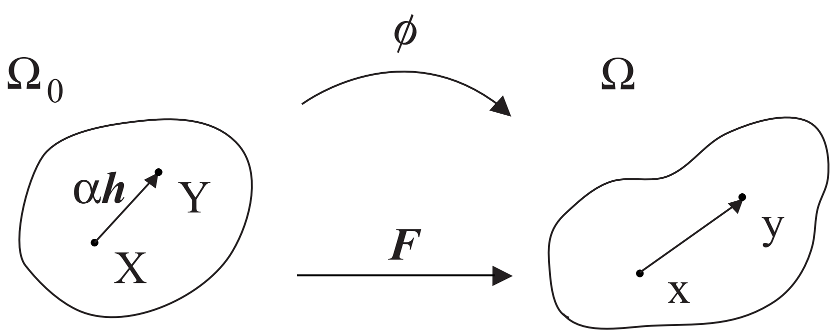

Figure 2: Mapping of vectors

Solution:

Let us define the vector between $\mathbf{X}$ and $\mathbf{Y}$ as,

$$\alpha\mathbf{h} \quad = \quad \mathbf{Y} - \mathbf{X} \, . \tag{41}$$

Then using Taylor series expansion with respect to α,

$$\begin{aligned} \mathbf{y} - \mathbf{x} &= \boldsymbol{\varphi}(\mathbf{Y}) - \boldsymbol{\varphi}(\mathbf{X}) \\ &= \boldsymbol{\varphi}(\mathbf{X} + \alpha\mathbf{h}) - \boldsymbol{\varphi}(\mathbf{X}) \\ &= \boldsymbol{\varphi}(\mathbf{X}) + \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{X}}\alpha\mathbf{h} + O(\alpha^2) - \boldsymbol{\varphi}(\mathbf{X}) \\ &= \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{X}}\alpha\mathbf{h} + O(\alpha^2) \\ &= \mathbf{F}\alpha\mathbf{h} + O(\alpha^2) \, . \end{aligned} \tag{42}$$

When the distance between $\mathbf{X}$ and $\mathbf{Y}$ is small, i.e. $\alpha << 1$, we can neglect $O(\alpha^2)$ compared to α, and,

$$\mathbf{y} - \mathbf{x} \quad \approx \quad \mathbf{F}\left(\mathbf{Y} - \mathbf{X}\right) \ . \tag{43}$$

From this we observe that, any vector in the material configuration is approximately mapped to the spatial configuration by the deformation gradient $\mathbf{F}$.

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Problem:

Give an interpretation of the right and left polar decomposition of the deformation gradient.

$$\mathbf{F} = \mathbf{RU} \tag{44}$$
$$= \mathbf{VR} \tag{45}$$

Solution:

Let us assume a hypothetical 2-D problem where we have a motion that maps a square having sides of unit length, to a slightly rotated rectangle with sides of λ. A schematic is shown in the figure below. The deformation gradient maps a

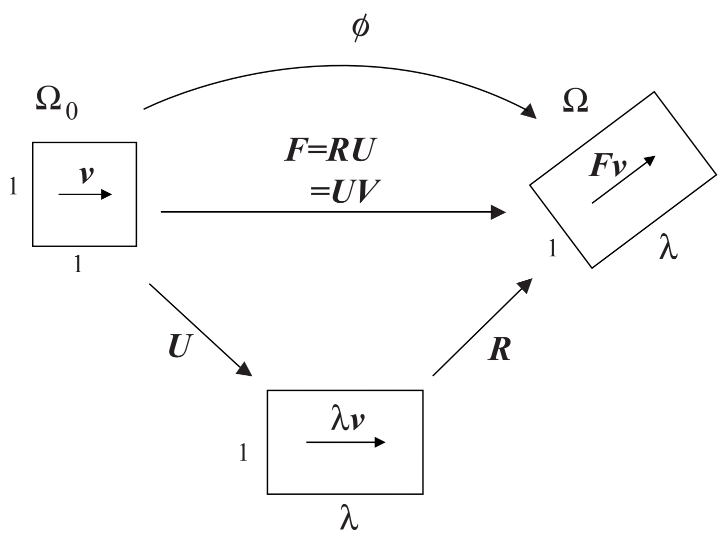

Figure 3: Interpretation of right polar decomposition

vector $\mathbf{v}$ to $\mathbf{Fv}$. If we consider the right polar decomposition, this mapping can be decomposed into two maps, first an application of $\mathbf{U}$, followed by $\mathbf{R}$. $\mathbf{U}$ simply stretches the square, then $\mathbf{R}$ rotates the body into the final configuration.

Alternatively, if we consider the left polar decomposition, this mapping can be decomposed into two maps, first an application of $\mathbf{R}$, followed by $\mathbf{V}$. $\mathbf{R}$ rotates the body into the proper orientation, then $\mathbf{V}$ simply stretches the square. This is depicted in the figure below.

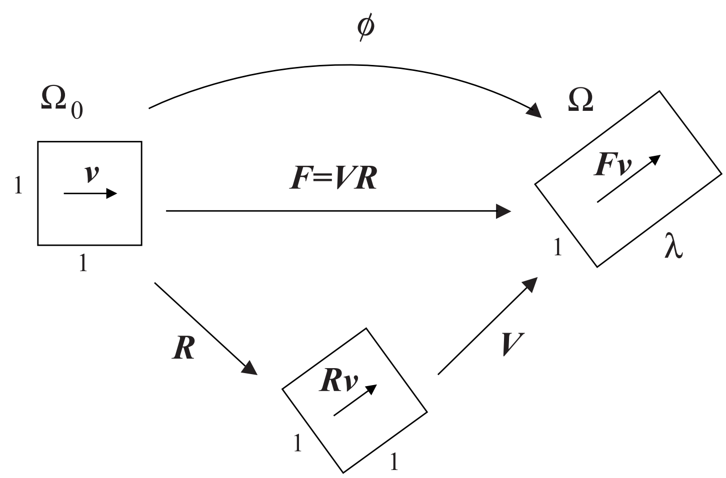

Figure 4: Interpretation of left polar decomposition

Problem:

Define the displacement vector $\mathbf{u}$ as,

$$\mathbf{u} = \mathbf{x} - \mathbf{X}\,. \tag{46}$$

Linearize the Green-Lagrange strain tensor,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{1}\right) \tag{47}$$

and Almansi strain tensor

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-T}\mathbf{F}{-1}\right) \tag{48}$$

around zero displacement and show that they match the infinitesimal strain tensor,

$$\varepsilon = \frac{1}{2}\left(\nabla_{\mathbf{X}}\mathbf{u} + (\nabla_{\mathbf{X}}\mathbf{u})^T\right) \tag{49}$$

Solution:

If we take a derivative of eqn. (46) with respect to $\mathbf{X}$,

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial \mathbf{X}} &= \frac{\partial \mathbf{x}}{\partial \mathbf{X}} - \mathbf{1} \\ \nabla_{\mathbf{X}}\mathbf{u} &= \mathbf{F} - \mathbf{1}\end{aligned} \tag{50}$$

Inserting this relationship into eqn. (47),

$$\mathbf{E} = \frac{1}{2}\left(\nabla_{\mathbf{X}}\mathbf{u} + (\nabla_{\mathbf{X}}\mathbf{u})^T + (\nabla_{\mathbf{X}}\mathbf{u})^T\,\nabla_{\mathbf{X}}\mathbf{u}\right)\,. \tag{51}$$

We look at $\mathbf{E}$ as a function with argument $\mathbf{u}$, i.e. $\mathbf{E}(\mathbf{u})$, and linearize at $\mathbf{0}$ in the direction of the displacment $\mathbf{h}$,

$$\begin{aligned} D\mathbf{E}(\mathbf{0})[\mathbf{h}] &= \left.\frac{d}{d\alpha}\right|_{\alpha=0} \mathbf{E}(\mathbf{0}+\alpha\mathbf{h}) \\ &= \left.\frac{d}{d\alpha}\right|_{\alpha=0} \frac{1}{2}\left(\alpha\nabla_{\mathbf{X}}\mathbf{h} + \alpha\,(\nabla_{\mathbf{X}}\mathbf{h})^T + \alpha^2\,(\nabla_{\mathbf{X}}\mathbf{h})^T\,\nabla_{\mathbf{X}}\mathbf{h}\right) \\ &= \frac{1}{2}\left(\nabla_{\mathbf{X}}\mathbf{h} + (\nabla_{\mathbf{X}}\mathbf{h})^T\right)\,. \end{aligned} \tag{52}$$

This shows that near $\mathbf{0}$, the Green-Lagrange strain tensor depends linearly on the displacement $\mathbf{h}$ in the same manner as the infinitesimal strain tensor.

Inserting eqn. (50) into relationship eqn. (48),

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{1} - (\mathbf{1}+\nabla_{\mathbf{X}}\mathbf{u})^{-T}\,(\mathbf{1}+\nabla_{\mathbf{X}}\mathbf{u})^{-1}\right)\,. \tag{53}$$

We look at $\mathbf{e}$ as a function with manner to $\mathbf{u}$,i.e. $\mathbf{e}(\mathbf{u})$, and linearize at $\mathbf{0}$ in the direction of the displacment $\mathbf{h}$,

$$\begin{aligned} D\mathbf{e}(\mathbf{0})[\mathbf{h}] &= \left.\frac{d}{d\alpha}\right|_{\alpha=0} \frac{1}{2}\left(\mathbf{1} - (\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h})^{-T}\,(\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h})^{-1}\right) \\ &= -\frac{1}{2}\left(\frac{d}{d\alpha}\,(\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h})^{-T}\right)(\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h})^{-1}\Big|_{\alpha=0} - (\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h})^{-T}\,\frac{1}{2}\left(\frac{d}{d\alpha}\,(\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h})^{-1}\right)\Big|_{\alpha=0} \end{aligned} \tag{54}$$

In order to compute this we must be able to evaluate,

$$\frac{d\mathbf{A}^{-1}(t)}{dt} \tag{55}$$

given a tensor valued function $\mathbf{A}(t)$ depending on a scalar variable t. We have the following relation between the tensor and its inverse,

$$\mathbf{A}(t)\mathbf{A}^{-1}(t) = \mathbf{1}\,. \tag{56}$$

If we take a derivative of both sides with respect to t we obtain,

$$\begin{aligned} \frac{\partial \mathbf{A}(t)}{\partial t}\mathbf{A}^{-1}(t) + \mathbf{A}\frac{\partial \mathbf{A}^{-1}(t)}{\partial t} &= \mathbf{0} \\ \frac{\partial \mathbf{A}(t)}{\partial t}\mathbf{A}^{-1}(t) &= -\mathbf{A}(t)\frac{\partial \mathbf{A}^{-1}(t)}{\partial t} \\ \frac{\partial \mathbf{A}(t)}{\partial t}\mathbf{A}^{-1}(t) &= -\mathbf{A}(t)\frac{\partial \mathbf{A}^{-1}(t)}{\partial t} \\ \frac{\partial \mathbf{A}^{-1}(t)}{\partial t} &= -\mathbf{A}^{-1}(t)\frac{\partial \mathbf{A}^{-1}(t)}{\partial t}\mathbf{A}^{-1}(t) \end{aligned} \tag{57}$$

Applying this to our problem we obtain,

$$\begin{aligned} \frac{d}{d\alpha}\left(\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h}\right)^{-1} &= -\left(\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h}\right)^{-1}\frac{d\left(\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h}\right)}{d\alpha}\left(\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h}\right)^{-1} \\ &= -\left(\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h}\right)^{-1}\nabla_{\mathbf{X}}\mathbf{h}\left(\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h}\right)^{-1}\ , \end{aligned} \tag{58}$$

and thus,

$$\left.\frac{d}{d\alpha}\right|_{\alpha=0}\left(\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h}\right)^{-1} = -\nabla_{\mathbf{X}}\mathbf{h} \tag{59}$$

$$\left.\frac{d}{d\alpha}\right|_{\alpha=0}\left(\mathbf{1}+\alpha\nabla_{\mathbf{X}}\mathbf{h}\right)^{-T} = -\nabla_{\mathbf{X}}\mathbf{h}^{T}\,. \tag{60}$$

Inserting this relationship into eqn. (54), we have,

$$\begin{aligned} De(\mathbf{0})[\mathbf{h}] &= -\frac{1}{2}\left(-\nabla_{\mathbf{X}}\mathbf{h}\right) - \frac{1}{2}\left(-\nabla_{\mathbf{X}}\mathbf{h}\right)^{T} \\ &= \frac{1}{2}\left(\nabla_{\mathbf{X}}\mathbf{h} + \left(\nabla_{\mathbf{X}}\mathbf{h}\right)^{T}\right)\,. \end{aligned} \tag{61}$$

This shows that near $\mathbf{0}$, the Almansi strain tensor depends linearly on the displacement $\mathbf{h}$ in the same manner as the infinitesimal strain tensor.

2 Homework

2.1 Computing material and spatial quantities

The motion for simple shear of a unit square ($\Omega = [0,1] \times [0,1]$) is given as,

$$x_1 = X_1 + X_2\gamma(t) \quad (62)$$
$$x_2 = X_2 \,. \quad (63)$$

Sketch the given motion, then compute the material velocity($\mathbf{V}(\mathbf{X},t)$), material acceleration($\mathbf{A}(\mathbf{X},t)$), spatial velocity($\mathbf{v}(\mathbf{x},t)$), spatial acceleration($\mathbf{a}(\mathbf{x},t)$), and confirm the following relationship between the material time derivative of the spatial velocity and the spatial time derivative of the spatial velocity,

$$\left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{X}} = \left(\frac{\partial \mathbf{v}}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)_t \cdot \mathbf{v} \,. \quad (64)$$

2.2 Deformations

Given the motion,

$$x_1 = \alpha \cos\theta X_1 - \beta \sin\theta X_2 + \gamma \quad (65)$$
$$x_2 = \alpha \sin\theta X_1 + \beta \cos\theta X_2 + \gamma \quad (66)$$

of a unit square ($\Omega = [0,1] \times [0,1]$) where α, β, γ are constants, compute the quantaties $\mathbf{F}, \mathbf{C}, \mathbf{b}, \mathbf{E}$ and $\mathbf{e}$. Then compute the right and left polar decompositions of $\mathbf{F}$,

$$\mathbf{F} = \mathbf{R}_U \mathbf{U} \quad (67)$$
$$\mathbf{F} = \mathbf{V}\mathbf{R}_V \quad (68)$$

and confirm that $\mathbf{R}_U = \mathbf{R}_V$. Lastly compute the spectral decompositions of $\mathbf{U}$ and $\mathbf{V}$,

$$\mathbf{U} = \Sigma_{A=1}^{2} \lambda_{UA} \mathbf{N}_A \otimes \mathbf{N}_A \quad (69)$$
$$\mathbf{V} = \Sigma_{A=1}^{2} \lambda_{VA} \mathbf{n}_A \otimes \mathbf{n}_A \quad (70)$$
$$(71)$$

and confirm that,

$$\lambda_{UA} = \lambda_{VA} \quad (72)$$
$$\mathbf{n}_A = \mathbf{R}\mathbf{N}_A \,. \quad (73)$$

Describe the given motion along with a sketch.