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Department of Mechanical and Process Engineering
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Nonlinear Continuum Mechanics
Exercise 3

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1 Useful Definitions or Concepts

1.1 Taking derivatives of functions

So far we have treated scalars, vectors, and 2nd-order tensors. Functions which take any of these types as arguments and return any of these types can be defined and derivatives of these functions can be obtained respective of their arguments, i.e. if the argument is a vector we can take a derivative with respect to the vector. Below we introduce the different notations that are possible.

1.1.1 Differentiating in the symbolic notation

The chart below shows the different combinations that ara available. For example, if the function is scalar-valued f and takes a vector $\mathbf{v}$ as an argument, we have,

$$\frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} \ . \tag{1}$$

Table 1: Differentiating in symbolic notation

		Values		
		Scalar (f)	Vector ($\mathbf{f}$)	Tensor ($\mathbf{F}$)
Args	Scalar t $\left(\frac{d}{dt}\right)$	$\frac{d}{dt}\{f(t)\}$	$\frac{d}{dt}\{\mathbf{f}(t))\}$	$\frac{d}{dt}\{\mathbf{F}(t)\}$
	Vector $\mathbf{v}$ $\left(\frac{\partial}{\partial \mathbf{v}}\right)$	$\frac{\partial}{\partial \mathbf{v}}\{f(\mathbf{v})\}$	$\frac{\partial}{\partial \mathbf{v}}\{\mathbf{f}(\mathbf{v})\}$	$\frac{\partial}{\partial \mathbf{v}}\{\mathbf{F}(\mathbf{v})\}$
	Tensor $\mathbf{A}$ $\left(\frac{\partial}{\partial \mathbf{A}}\right)$	$\frac{\partial\{f(\mathbf{A})\}}{\partial \mathbf{A}}$	$\frac{\partial\{\mathbf{f}(\mathbf{A})\}}{\partial \mathbf{A}}$	$\frac{\partial\{\mathbf{F}(\mathbf{A})\}}{\partial \mathbf{A}}$

1.1.2 Differentiating in index notation

When one would like to manipulate the components for calculation, the differentiation of the functions can be expressed in index notation. For example, if the function is scalar-valued f and takes a tensor $\mathbf{A}$ as an argument, we have,

$$\frac{\partial f(\mathbf{A})}{\partial \mathbf{A}} = \frac{\partial f(\mathbf{A})}{\partial A_{kl}} \mathbf{e}_k \otimes \mathbf{e}_l \tag{2}$$

One must take care in appending the basis vectors $\mathbf{e}_i$ with indices related to differentiation to the end, NOT THE FRONT. (Though this is a convention and different books may define the differentiation operations differently).

Table 2: Differentiating in index notation

		Values		
		Scalar (f)	Vector ($\mathbf{f} = f_i \mathbf{e}_i$)	Tensor ($\mathbf{F} = F_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$)
Args	Scalar t $\left(\frac{d}{dt}\right)$	$\frac{d}{dt}\{f(t)\}$	$\frac{d}{dt}\{f_i(t)\mathbf{e}_i\}$	$\frac{d}{dt}\{F_{ij}(t)\mathbf{e}_i \otimes \mathbf{e}_j\}$
	Vector $\mathbf{v}$ $\left(\frac{\partial}{\partial v_k}\mathbf{e}_k\right)$	$\frac{\partial}{\partial v_k}\{f(\mathbf{v})\}\mathbf{e}_k$	$\frac{\partial}{\partial v_k}\{f_i(\mathbf{v})\mathbf{e}_i\} \otimes \mathbf{e}_k$	$\frac{\partial}{\partial v_k}\{F_{ij}(\mathbf{v})\mathbf{e}_i \otimes \mathbf{e}_j\} \otimes \mathbf{e}_k$
	Tensor $\mathbf{A}$ $\left(\frac{\partial}{\partial A_{kl}}\mathbf{e}_k \otimes \mathbf{e}_l\right)$	$\frac{\partial\{f(\mathbf{A})\}}{\partial A_{kl}}\mathbf{e}_k \otimes \mathbf{e}_l$	$\frac{\partial\{f_i(\mathbf{A})\mathbf{e}_i\}}{\partial A_{kl}} \otimes \mathbf{e}_k \otimes \mathbf{e}_l$	$\frac{\partial\{F_{ij}(\mathbf{A})\mathbf{e}_i \otimes \mathbf{e}_j\}}{\partial A_{kl}} \otimes \mathbf{e}_k \otimes \mathbf{e}_l$

1.1.3 Differentiating in simplified index notation

When one would like to manipulate the components for calculation, the differentiation of the functions can be expressed in index notation. Additionally when it is clear what basis are being used consistently for all objects, we abreivate the basis for simpler notation. For example, if the function is vector-valued $\mathbf{f}$ and takes a vector $\mathbf{v}$ as an argument, we have,

$$\begin{aligned} \frac{\partial f(\mathbf{f})}{\partial \mathbf{v}} &= \frac{\partial f_i(\mathbf{v})\mathbf{e}_i}{\partial v_k} \otimes \mathbf{e}_k \\ &= \frac{\partial f_i(\mathbf{v})}{\partial v_k}\mathbf{e}_i \otimes \mathbf{e}_k \\ &\text{In simplified index notation } \frac{\partial f_i(\mathbf{v})}{\partial v_k} \end{aligned} \tag{3}$$

One must take care in appending the basis vectors $\mathbf{e}_i$ with indices related to differentiation to the end, NOT THE FRONT. (Though this is a convention and different books may define the differentiation operations differently).

Table 3: Differentiating in simplified index notation

		Values		
		Scalar (f)	Vector ($\mathbf{f} = f_i$)	Tensor ($\mathbf{F} = F_{ij}$)
Args	Scalar t $\left(\frac{d}{dt}\right)$	$\frac{d}{dt}\{f(t)\}$	$\frac{d}{dt}\{f_i(t)\}$	$\frac{d}{dt}\{F_{ij}(t)\}$
	Vector $\mathbf{v}$ $\left(\frac{\partial}{\partial v_k}\right)$	$\frac{\partial}{\partial v_k}\{f(\mathbf{v})\}$	$\frac{\partial}{\partial v_k}\{f_i(\mathbf{v})\}$	$\frac{\partial}{\partial v_k}\{F_{ij}(\mathbf{v})\}$
	Tensor $\mathbf{A}$ $\left(\frac{\partial}{\partial A_{kl}}\right)$	$\frac{\partial\{f(\mathbf{A})\}}{\partial A_{kl}}$	$\frac{\partial\{f_i(\mathbf{A})\}}{\partial A_{kl}}$	$\frac{\partial\{F_{ij}(\mathbf{A})\}}{\partial A_{kl}}$

In the special case when the argument is dependent on the coordinates $\mathbf{x}$, then we have an even simpler notation. notation. For example, if the function is vector-valued $\mathbf{f}$ and takes the coordinates $\mathbf{x}$ as an argument, we have,

$$\begin{aligned} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} &= \frac{\partial f_i(\mathbf{x})\mathbf{e}_i}{\partial x_k} \otimes \mathbf{e}_k \\ &= \frac{\partial f_i(\mathbf{x})}{\partial x_k}\mathbf{e}_i \otimes \mathbf{e}_k \\ &\text{In simplified index notation } \frac{\partial f_i(\mathbf{x})}{\partial x_k} \\ &\text{In simplified index notation } f_{i,k}(\mathbf{x}) \end{aligned} \tag{4}$$

One must take care in appending the basis vectors $\mathbf{e}_i$ with indices related to differentiation to the end, NOT THE FRONT. (Though this is a convention and different books may define the differentiation operations differently).

Table 4: Differentiating in simplified index notation(coordinate arguments)

		Values		
		Scalar (f)	Vector ($\mathbf{f} = f_i$)	Tensor ($\mathbf{F} = F_{ij}$)
Args	Vector $\mathbf{x}$ $\left(\frac{\partial}{\partial x_k}\right)$	$f_k(\mathbf{x})$	$f_{i,k}(\mathbf{x})$	$F_{ij,k}(\mathbf{x})$

1.2 The directional derivative

The directional derivative of a function f with argument v in direction h is defined as,

$$Df(v)[h] = \frac{d}{d\alpha}|_{\alpha=0} f(v+\alpha h) \quad (5)$$

$$= \lim_{\alpha \to 0} \frac{f(v+\alpha h) - f(v)}{\alpha} \quad (6)$$

The expressions on the right hand side are equivalent from the definition of the derivative. This directional derivative is useful in computing the derivative of a function f (scalar, vector, tensor valued), since we have the following relations between the two.

$$Df(t)[h] = \frac{df(t)}{dt} h \quad (7)$$

$$Df(\mathbf{v})[\mathbf{h}] = \frac{df(\mathbf{v})}{d\mathbf{v}} \cdot \mathbf{h} \quad (8)$$

$$Df(\mathbf{A})[\mathbf{H}] = \frac{df(\mathbf{A})}{d\mathbf{A}} : \mathbf{H} \quad (9)$$

1.3 Computing derivatives

To compute the derivative of a function one has two approaches.

1. Compute the directional derivative and extract the derivative.
2. Compute the derivative directly through simplified index calculation.

Depending on the problem one approach may be quicker and easier than the other. To compute the derivative of the trace function we have the following two approaches.

1. Compute $D\text{tr}(\mathbf{A})[\mathbf{H}]$.
2. Compute $\frac{\partial \text{tr}(\mathbf{A})}{\partial \mathbf{A}} = \frac{\partial A_{ii}}{\partial A_{kl}}$.

1.4 Important operators

1.4.1 The gradient operator

Using the operator,

$$\nabla = \frac{\partial}{\partial \mathbf{x}} = \frac{\partial}{\partial x_k}\mathbf{e}_k \tag{10}$$

one can define the gradient operation on scalar, vector, and tensor fields.

$$\begin{aligned}
\mathrm{grad} f(\mathbf{x}) &= \nabla f(\mathbf{x}) \\
&= \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \\
&= \frac{\partial f(\mathbf{x})}{\partial x_k}\mathbf{e}_k \\
&\quad \text{In simplified index notation } \frac{\partial f(\mathbf{x})}{\partial x_k} = f_{,k} \\
\mathrm{grad}\mathbf{v}(\mathbf{x}) &= \nabla \otimes \mathbf{v}(\mathbf{x}) \\
&= \frac{\partial \mathbf{v}(\mathbf{x})}{\partial \mathbf{x}} \\
&= \frac{\partial v_i(\mathbf{x})\mathbf{e}_i}{\partial x_k} \otimes \mathbf{e}_k = \frac{\partial v_i(\mathbf{x})}{\partial x_k}\mathbf{e}_i \otimes \mathbf{e}_k \\
&\quad \text{In simplified index notation } \frac{\partial v_i(\mathbf{x})}{\partial x_k} = v_{i,k} \\
\mathrm{grad}\mathbf{A}(\mathbf{x}) &= \nabla \otimes \mathbf{A}(\mathbf{x}) \\
&= \frac{\partial \mathbf{A}(\mathbf{x})}{\partial \mathbf{x}} \\
&= \frac{\partial A_{ij}(\mathbf{x})\mathbf{e}_i \otimes \mathbf{e}_j}{\partial x_k} \otimes \mathbf{e}_k = \frac{\partial A_{ij}(\mathbf{x})}{\partial x_k}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \\
&\quad \text{In simplified index notation } \frac{\partial A_{ij}(\mathbf{x})}{\partial x_k} = A_{ij,k}
\end{aligned} \tag{11}$$

1.4.2 The divergence operator

Using the operator,

$$\nabla = \frac{\partial}{\partial \mathbf{x}} = \frac{\partial}{\partial x_k}\mathbf{e}_k \tag{12}$$

one can define the divergence operation on vector and tensor fields.

$$\begin{aligned}
\text{div}\mathbf{v}(\mathbf{x}) &= \nabla \cdot \mathbf{v}(\mathbf{x}) \\
&= \frac{\partial v_i(\mathbf{x})\mathbf{e}_i}{\partial x_k} \cdot \mathbf{e}_k = \frac{\partial v_i(\mathbf{x})}{\partial x_k}\delta_{ik} \\
&= \frac{\partial v_i(\mathbf{x})}{\partial x_i} \\
&= v_{i,i} \\
\text{div}\mathbf{A}(\mathbf{x}) &= \nabla \cdot \mathbf{A}(\mathbf{x}) \\
&= \frac{\partial A_{ij}(\mathbf{x})\mathbf{e}_i \otimes \mathbf{e}_j}{\partial x_k} \cdot \mathbf{e}_k = \frac{\partial A_{ij}(\mathbf{x})}{\partial x_k}\delta_{jk}\mathbf{e}_i \\
&= \frac{\partial A_{ij}(\mathbf{x})}{\partial x_j}\mathbf{e}_i
\end{aligned}$$

In simplified index notation $\frac{\partial A_{ij}(\mathbf{x})}{\partial x_j} = A_{ij,j}$

1.4.3 The curl operator

Using the operator,

$$\nabla = \frac{\partial}{\partial \mathbf{x}} = \frac{\partial}{\partial x_k}\mathbf{e}_k \tag{13}$$

one can define the curl operation on a vector field,

$$\begin{aligned}
\text{curl}\mathbf{v}(\mathbf{x}) &= \nabla \times \mathbf{v}(\mathbf{x}) \\
&= \mathbf{e}_k \times \frac{\partial v_i(\mathbf{x})\mathbf{e}_i}{\partial x_k} \\
&= \frac{\partial v_i(\mathbf{x})}{\partial x_k}\mathbf{e}_k \otimes \mathbf{e}_i \\
&= \frac{\partial v_i(\mathbf{x})}{\partial x_k}\varepsilon_{kij}\mathbf{e}_j
\end{aligned}$$

In simplified index notation $\frac{\partial v_i(\mathbf{x})}{\partial x_k}\varepsilon_{kij} = v_{i,k}\varepsilon_{kij}$

1.5 Integral theorem

Given any field,

- A scalar T
- A vector T_i
- A 2nd-order tensor T_{ij}
- A kth-order tensor $T_{ij\cdots}$

we can apply the following integral theorem which converts volume integration to surface integration.

$$\int_\Omega T_{ij\cdots p,q}dV = \int_{\partial\Omega} T_{ij\cdots p}n_q dA \tag{14}$$

Here n_q is the normal vector to the surface $\partial\Omega$.

2 Application of concepts or definitions

Problem:

Show that the relationship,

$$Df(t)[h] = \frac{df(t)}{dt}h \tag{15}$$

$$\tag{16}$$

is valid for a scalar valued scalar argument function.

Solution:

From the Taylor expansion of $f(t + \alpha h)$ at t we have,

$$f(t + \alpha h) = f(t) + \frac{df}{dt}(t)\alpha h + O(\alpha^2) .$$

Manipulating this we obtain,

$$\frac{f(t + \alpha h) - f(t)}{\alpha} = \frac{df}{dt}(t)h + O(\alpha) .$$

From the definition of the directional derivative we have,

$$\begin{aligned} Df(t)[h] &= \lim_{\alpha \to 0} \frac{f(t + \alpha h) - f(t)}{\alpha} \\ &= \lim_{\alpha \to 0} \frac{df}{dt}(t)h + O(\alpha) \\ &= \frac{df}{dt}(t)h . \end{aligned}$$

Problem:

Show that the relationship,

$$Df(\mathbf{v})[\mathbf{h}] = \frac{\partial f(\mathbf{v})}{\partial \mathbf{v}} \cdot \mathbf{h} \tag{17}$$

$$\tag{18}$$

is valid for a scalar valued vector argument function.

Solution:

From the Taylor expansion of $f(t + \alpha h)$ at t we have,

$$\begin{aligned} f(\mathbf{v} + \alpha \mathbf{h}) &= f(v_1 + \alpha h_1, \ldots, v_n + \alpha h_n) \\ &= f(v_1, \ldots, v_n) + \frac{\partial f}{\partial v_1}(\mathbf{v})\alpha h_1 + \cdots + \frac{\partial f}{\partial v_n}(\mathbf{v})\alpha h_n + O(\alpha^2) . \\ &= f(\mathbf{v}) + \frac{\partial f}{\partial v_i}(\mathbf{v})\alpha h_i + O(\alpha^2) . \\ &= f(\mathbf{v}) + \alpha \frac{\partial f}{\partial \mathbf{v}}(\mathbf{v}) \cdot \mathbf{h} + O(\alpha^2) . \end{aligned}$$

Manipulating this we obtain,

$$\frac{f(\mathbf{v}+\alpha\mathbf{h}-f(\mathbf{v})}{\alpha} = \frac{\partial f}{\partial \mathbf{v}}(\mathbf{v})\cdot\mathbf{h}+O(\alpha)\ .$$

From the definition of the directional derivative we have,

$$\begin{aligned} Df(\mathbf{v})[\mathbf{h}] &= \lim_{\alpha\to 0}\frac{f(\mathbf{v}+\alpha\mathbf{h})-f(\mathbf{v})}{\alpha} \\ &= \lim_{\alpha\to 0}\frac{\partial f}{\partial \mathbf{v}}(\mathbf{v})\cdot\mathbf{h}+O(\alpha) \\ &= \frac{\partial f}{\partial \mathbf{v}}(\mathbf{v})\mathbf{h}\ . \end{aligned}$$

Problem:

Show that the relationship,

$$Df(\mathbf{A})[\mathbf{H}] = \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}}:\mathbf{H} \tag{19}$$

is valid for a scalar valued tensor argument function.

Solution:

From the Taylor expansion of $f(t+\alpha h)$ at t we have,

$$\begin{aligned} f(\mathbf{A}+\alpha\mathbf{H}) &= f(A_{11}+\alpha H_{11},\ldots,A_{nn}+\alpha H_{nn}) \\ &= f(A_{11},\ldots,H_{nn})+\frac{\partial f}{\partial A_{11}}(\mathbf{A})\alpha H_{11}+\cdots+\frac{\partial f}{\partial A_{nn}}(\mathbf{A})\alpha H_{nn}+O(\alpha^2)\ . \\ &= f(\mathbf{A})+\frac{\partial f}{\partial A_{ij}}(\mathbf{v})\alpha H_{ij}+O(\alpha^2)\ . \\ &= f(\mathbf{v})+\alpha\frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}):\mathbf{H}+O(\alpha^2)\ . \end{aligned}$$

Manipulating this we obtain,

$$\frac{f(\mathbf{A}+\alpha\mathbf{H}-f(\mathbf{A})}{\alpha} = \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}):\mathbf{H}+O(\alpha)\ .$$

From the definition of the directional derivative we have,

$$\begin{aligned} Df(\mathbf{A})[\mathbf{H}] &= \lim_{\alpha\to 0}\frac{f(\mathbf{A}+\alpha\mathbf{H})-f(\mathbf{A})}{\alpha} \\ &= \lim_{\alpha\to 0}\frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}):\mathbf{H}+O(\alpha) \\ &= \frac{\partial f}{\partial \mathbf{A}}(\mathbf{A}):\mathbf{H}\ . \end{aligned}$$

Problem:

Find the derivative of the scalar valued tensor argument function,

$$\mathrm{tr}(\mathbf{A}) \tag{20}$$

Solution:

Taking the directional derivative,

$$\begin{aligned} \frac{\partial \text{tr}(\mathbf{A})}{\partial \mathbf{A}} : \mathbf{H} &= \frac{d}{d\alpha} |_{\alpha=0} \text{tr}(\mathbf{A} + \alpha \mathbf{H}) \\ &= \frac{df}{d\alpha} |_{\alpha=0} \text{tr}(\mathbf{A}) + \alpha \text{tr}(\mathbf{H}) \qquad (21) \\ &= \text{tr}(\mathbf{H}) \\ &= \mathbf{1} : \mathbf{H} \qquad (22) \end{aligned}$$

Thus we have that,

$$\frac{\partial \text{tr}(\mathbf{A})}{\partial \mathbf{A}} = \mathbf{1} \qquad (23)$$

Alternatively, we can compute this derivative through the simplified index notation.

$$\begin{aligned} \left[\frac{\partial \text{tr}(\mathbf{A})}{\partial \mathbf{A}} \right]_{kl} &= \frac{\partial \text{tr}(\mathbf{A})}{\partial A_{kl}} \\ &= \frac{\partial A_{ii}}{\partial A_{kl}} \\ &= \delta_{ik} \delta_{il} \\ &= \delta_{kl} \qquad (24) \end{aligned}$$

Thus we again have the relation in equation (23).

Problem:

Find the derivative of the scalar valued vector argument function,

$$||\mathbf{v}|| \qquad (25)$$

Solution:

Taking the directional derivative,

$$\begin{aligned} \frac{\partial ||\mathbf{v}||}{\partial \mathbf{v}} \cdot \mathbf{h} &= \frac{d}{d\alpha} |_{\alpha=0} \sqrt{(\mathbf{v} + \alpha \mathbf{h}) \cdot (\mathbf{v} + \alpha \mathbf{h})} \\ &= \frac{\mathbf{h} \cdot (\mathbf{v} + \alpha \mathbf{h}) + (\mathbf{v} + \alpha \mathbf{h}) \cdot \mathbf{h}}{2\sqrt{(\mathbf{v} + \alpha \mathbf{h}) \cdot (\mathbf{v} + \alpha \mathbf{h})}} |_{\alpha=0} \\ &= \frac{\mathbf{v} \cdot \mathbf{h}}{||\mathbf{v}||} \\ &= \frac{\mathbf{v}}{||\mathbf{v}||} \cdot \mathbf{h} \end{aligned} \qquad (26)$$

Thus we have that,

$$\frac{\partial ||\mathbf{v}||}{\partial \mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||} \qquad (27)$$

Alternatively, we can compute this derivative through the simplified index notation.

$$\begin{aligned}\left[\frac{\partial||\mathbf{v}||}{\partial\mathbf{v}}\right]_k &= \frac{\partial\sqrt{v_iv_i}}{\partial v_k}\\ &= \frac{1}{2\sqrt{v_jv_j}}\frac{\partial v_iv_i}{\partial v_k}\\ &= \frac{1}{2||\mathbf{v}||}\left(\delta_{ik}v_i + v_i\delta_{ik}\right)\\ &= \frac{1}{2||\mathbf{v}||}\left(2v_k\right)\\ &= \frac{v_k}{||\mathbf{v}||}\end{aligned} \tag{28}$$

Thus we again have the relation in equation (27).

Problem:

Find the derivative of the scalar valued vector argument function,

$$f(\mathbf{v}) = \mathbf{a}\cdot\mathbf{v} \tag{29}$$

where $\mathbf{a}$ is a constant vector.

Solution:

Taking the directional derivative,

$$\begin{aligned}\frac{\partial\mathbf{a}\cdot\mathbf{v}}{\partial\mathbf{v}}\cdot\mathbf{h}\\ &= \left.\frac{d}{d\alpha}\right|_{\alpha=0}\mathbf{a}\cdot(\mathbf{v}+\alpha\mathbf{h})\\ &= \left.\mathbf{a}\cdot\mathbf{h}\right|_{\alpha=0}\\ &= \mathbf{a}\cdot\mathbf{h}\end{aligned} \tag{30}$$

Thus we have that,

$$\frac{\partial\mathbf{a}\cdot\mathbf{v}}{\partial\mathbf{v}} = \mathbf{a} \tag{31}$$

Alternatively, we can compute this derivative through the simplified index notation.

$$\begin{aligned}\left[\frac{\partial\mathbf{a}\cdot\mathbf{v}}{\partial\mathbf{v}}\right]_k &= \frac{\partial a_iv_i}{\partial v_k}\\ &= a_i\delta_{ik}\\ &= a_k\end{aligned} \tag{32}$$

Thus we again have the relation in equation (31).

Problem:

Find the derivative of the scalar valued vector argument function,

$$f(\mathbf{v}) \quad = \quad \mathbf{v} \cdot \mathbf{v} \tag{33}$$

Solution:

Taking the directional derivative,

$$\begin{aligned} \frac{\partial \mathbf{v} \cdot \mathbf{v}}{\partial \mathbf{v}} \cdot \mathbf{h} & \\ &= \left.\frac{d}{d\alpha}\right|_{\alpha=0} (\mathbf{v} + \alpha \mathbf{h}) \cdot (\mathbf{v} + \alpha \mathbf{h}) \\ &= \left.\mathbf{h} \cdot (\mathbf{v} + \alpha \mathbf{h}) + (\mathbf{v} + \alpha \mathbf{h}) \cdot \mathbf{h}\right|_{\alpha=0} \\ &= 2\mathbf{v} \cdot \mathbf{h} \end{aligned} \tag{34}$$

Thus we have that,

$$\frac{\partial \mathbf{v} \cdot \mathbf{v}}{\partial \mathbf{v}} = \mathbf{a} \tag{35}$$

Alternatively, we can compute this derivative through the simplified index notation.

$$\begin{aligned} \left[\frac{\partial \mathbf{v} \cdot \mathbf{v}}{\partial \mathbf{v}}\right]_k &= \frac{\partial v_i v_i}{\partial v_k} \\ &= \delta_{ik} v_i + v_i \delta_{ik} \\ &= 2 v_k \end{aligned} \tag{36}$$

Thus we again have the relation in equation (35).

Problem:

Find the derivative of the vector valued vector argument function,

$$f(\mathbf{v}) \quad = \quad \mathbf{v} \tag{37}$$

Solution:

Taking the directional derivative,

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial \mathbf{v}} \cdot \mathbf{h} & \\ &= \left.\frac{d}{d\alpha}\right|_{\alpha=0} (\mathbf{v} + \alpha \mathbf{h}) \\ &= \left.(\mathbf{v} + \alpha \mathbf{h})\right|_{\alpha=0} \\ &= \mathbf{h} \\ &= \mathbf{1h} \end{aligned} \tag{38}$$

Thus we have that,

$$\frac{\partial \mathbf{v}}{\partial \mathbf{v}} = \mathbf{1} \tag{39}$$

Alternatively, we can compute this derivative through the simplified index notation.

$$\begin{aligned} \left[\frac{\partial \mathbf{v}}{\partial \mathbf{v}}\right]_{ik} &= \frac{\partial v_i}{\partial v_k} \\ &= \delta_{ik} \end{aligned} \tag{40}$$

Thus we again have the relation in equation (39).

Problem:

Find the derivative of the vector valued vector argument function,

$$f(\mathbf{v}) = \mathbf{A}\mathbf{v} \tag{41}$$

where $\mathbf{A}$ is a constant tensor.

Solution:

Taking the directional derivative,

$$\begin{aligned} \frac{\partial \mathbf{A}\mathbf{v}}{\partial \mathbf{v}} \cdot \mathbf{h} & \\ &= \frac{d}{d\alpha}\bigg|_{\alpha=0} \mathbf{A}(\mathbf{v} + \alpha \mathbf{h}) \\ &= \mathbf{A}\mathbf{h}\bigg|_{\alpha=0} \\ &= \mathbf{A}\mathbf{h} \end{aligned} \tag{42}$$

Thus we have that,

$$\frac{\partial \mathbf{A}\mathbf{v}}{\partial \mathbf{v}} = \mathbf{A} \tag{43}$$

Alternatively, we can compute this derivative through the simplified index notation.

$$\begin{aligned} \left[\frac{\partial \mathbf{A}\mathbf{v}}{\partial \mathbf{v}}\right]_{ik} &= \frac{\partial A_{ij} v_j}{\partial v_k} \\ &= A_{ij}\delta_{jk} \\ &= A_{ik} \end{aligned} \tag{44}$$

Thus we again have the relation in equation (43).

Problem:

Find the derivative of the scalar valued tensor argument function,

$$f(\mathbf{A}) = \mathbf{A} : \mathbf{B} \tag{45}$$

where $\mathbf{B}$ is a constant tensor.

Solution:

Taking the directional derivative,

$$\begin{aligned}\frac{\partial \mathbf{A}:\mathbf{B}}{\partial \mathbf{A}}:\mathbf{H} &= \left.\frac{d}{d\alpha}\right|_{\alpha=0}(\mathbf{A}+\alpha\mathbf{H}):\mathbf{B} \\ &= \left.\mathbf{H}:\mathbf{B}\right|_{\alpha=0} \\ &= \mathbf{B}:\mathbf{H} \qquad (46)\end{aligned}$$

Thus we have that,

$$\frac{\partial \mathbf{A}:\mathbf{B}}{\partial \mathbf{A}} = \mathbf{B} \qquad (47)$$

Alternatively, we can compute this derivative through the simplified index notation.

$$\begin{aligned}\left[\frac{\partial \mathbf{A}:\mathbf{B}}{\partial \mathbf{A}}\right]_{kl} &= \frac{\partial A_{ij}B_{ij}}{\partial A_{kl}} \\ &= B_{ij}\delta_{ik}\delta_{jl} \\ &= A_{kl} \qquad (48)\end{aligned}$$

Thus we again have the relation in equation (47).

Problem:

Find the derivative of the scalar valued tensor argument function,

$$f(\mathbf{A}) = \mathbf{A}:\mathbf{A} \qquad (49)$$

Solution:

Taking the directional derivative,

$$\begin{aligned}\frac{\partial \mathbf{A}:\mathbf{A}}{\partial \mathbf{A}}:\mathbf{H} &= \left.\frac{d}{d\alpha}\right|_{\alpha=0}(\mathbf{A}+\alpha\mathbf{H}):(\mathbf{A}+\alpha\mathbf{H}) \\ &= \left.\mathbf{H}:(\mathbf{A}+\alpha\mathbf{H})+(\mathbf{A}+\alpha\mathbf{H}):\mathbf{H}\right|_{\alpha=0} \\ &= 2\mathbf{A}:\mathbf{H} \qquad (50)\end{aligned}$$

Thus we have that,

$$\frac{\partial \mathbf{A}:\mathbf{A}}{\partial \mathbf{A}} = 2\mathbf{A} \qquad (51)$$

Alternatively, we can compute this derivative through the simplified index notation.

$$\begin{aligned}\left[\frac{\partial \mathbf{A}:\mathbf{A}}{\partial \mathbf{A}}\right]_{kl} &= \frac{\partial A_{ij}A_{ij}}{\partial A_{kl}} \\ &= \delta_{ik}\delta_{jl}A_{ij}+A_{ij}\delta_{ik}\delta_{jl} \\ &= 2A_{kl} \qquad (52)\end{aligned}$$

Thus we again have the relation in equation (51).

Problem:

Show that the following relationship holds.

$$\int_{\partial\Omega} \Psi \mathbf{u} \cdot \mathbf{n} dA = \int_{\Omega} \text{div}(\Psi \mathbf{u}) dV \tag{53}$$

Solution:

$$\begin{aligned} \int_{\partial\Omega} \Psi \mathbf{u} \cdot \mathbf{n} dA &= \int_{\partial\Omega} \Psi u_i n_i dA \\ &= \int_{\Omega} (\Psi u_i)_{,i} \, dV \\ &= \int_{\Omega} \text{div} \, (\Psi \mathbf{u}) \, dV \end{aligned} \tag{54}$$

Problem:

Show that the following relationship holds.

$$\int_{\partial\Omega} \Psi \mathbf{A} \mathbf{n} dA = \int_{\Omega} \text{div}(\Psi \mathbf{A}) dV \tag{55}$$

Solution:

$$\begin{aligned} \left[\int_{\partial\Omega} \Psi \mathbf{A} \mathbf{n} dA \right]_i &= \int_{\partial\Omega} \Psi A_{ij} n_j dA \\ &= \int_{\Omega} (\Psi A_{ij})_{,j} \, dV \\ &= \left[\int_{\Omega} \text{div} \, (\Psi \mathbf{A}) \, dV \right]_i \end{aligned} \tag{56}$$

Problem:

Show that the following relationship holds.

$$\int_{\partial\Omega} \mathbf{u} \cdot \mathbf{A} \mathbf{n} dA = \int_{\Omega} \text{div}(\mathbf{A}^T \mathbf{u}) dV \tag{57}$$

Solution:

$$\begin{aligned} \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{A} \mathbf{n} dA &= \int_{\partial\Omega} u_i A_{ij} n_j dA \\ &= \int_{\Omega} (u_i A_{ij})_{,j} \, dV \\ &= \int_{\Omega} \left((\mathbf{A})^T_{ji} \, u_i \right)_{,j} \, dV \\ &= \int_{\Omega} \left(\mathbf{A}^T \mathbf{u} \right)_{j,j} dV \\ &= \int_{\Omega} \text{div} \left(\mathbf{A}^T \mathbf{u} \right) dV \end{aligned} \tag{58}$$

Problem:

Show that the following relationship holds.

$$\text{div}\,(\Psi\mathbf{u}) = \Psi\text{div}\,(\mathbf{u}) + \mathbf{u}\cdot\text{grad}\Psi \tag{59}$$

Solution:

$$\begin{aligned}\text{div}\,(\Psi\mathbf{u}) &= (\Psi u_i\mathbf{e}_i)_{,j}\otimes\mathbf{e}_j \\ &= (\Psi u_i)_{,i} \\ &= \Psi_{,i}u_i + \Psi u_{i,i} \\ &= \text{grad}\Psi\cdot\mathbf{u} + \Psi\text{div}\mathbf{u}\end{aligned} \tag{60}$$

Problem:

Show that the following relationship holds.

$$\text{div}\,(\Psi\mathbf{A}) = \Psi\text{div}\,(\mathbf{A}) + \mathbf{A}\text{grad}\Psi \tag{61}$$

Solution:

$$\begin{aligned}[\text{div}\,(\Psi\mathbf{A})]_i &= [\text{div}\,(\Psi\mathbf{A})]\cdot\mathbf{e}_i \\ &= \left[(\Psi A_{kl}\mathbf{e}_k\otimes\mathbf{e}_l)_{,j}\cdot\mathbf{e}_j\right]\cdot\mathbf{e}_i \\ &= \left[(\Psi A_{kl})_{,j}\,\delta_{lj}\mathbf{e}_k\right]\cdot\mathbf{e}_i \\ &= (\Psi A_{kl})_{,j}\,\delta_{lj}\delta_{ki} \\ &= (\Psi A_{ij})_{,j} \\ &= \Psi_{,j}A_{ij} + \Psi A_{ij,j} \\ &= [\mathbf{A}\text{grad}\Psi]_i + [\Psi\text{div}\mathbf{A}]_i\end{aligned} \tag{62}$$

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3 Homework

3.1 The skew tensor and its axial vector

Given a 2nd-order skew tensor $\mathbf{W}$, show that there exists a vector $\boldsymbol{\omega} \in \mathbb{R}^3$ such that,

$$\mathbf{W}\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v} \qquad \forall \mathbf{v} \in \mathbb{R}^3 \, . \tag{63}$$

This vector $\boldsymbol{\omega}$ is called the axial vector of $\mathbf{W}$. Express the components of $\boldsymbol{\omega}$ in terms of the components of $\mathbf{W}$. (Hint: Think of how many independent components the skew tensor has. Does this match the number of components of a vector in $\mathbb{R}^3$?)

Solution:

In components, a skew symmetric tensor $\mathbf{W}$ can be written as the matrix,

$$\mathbf{W} = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ \beta & -\gamma & 0 \end{bmatrix} . \tag{64}$$

It is clear that this has only three independent degrees of freedom, which matches that of a vector in $\mathbb{R}^3$. Given an arbitrary vector,

$$\mathbf{v} = [v_1 \, , v_2 \, , v_3 \,]^T \tag{65}$$

we have,

$$\mathbf{W}\mathbf{v} = \begin{bmatrix} \alpha v_2 + \beta v_3 \\ -\alpha v_1 + \gamma v_3 \\ -\beta v_1 - \gamma v_2 \end{bmatrix} \tag{66}$$

$$\boldsymbol{\omega} \times \mathbf{v} = \begin{bmatrix} \omega_2 v_3 - \omega_3 v_2 \\ \omega_3 v_1 - \omega_1 v_3 \\ \omega_1 v_2 - \omega_2 v_1 \end{bmatrix} . \tag{67}$$

Since these expressions must be equal for any v_i $(i = 1, 2, 3)$ we have that,

$$\omega_1 = -\gamma = W_{32} = -W_{23} \tag{68}$$

$$\omega_2 = \beta = W_{13} = -W_{31} \tag{69}$$

$$\omega_3 = -\alpha = W_{21} = -W_{12} \, . \tag{70}$$

Alternatvely we can obtain the desired result as follows.

$$\begin{aligned} \mathbf{W}\mathbf{v} &= \boldsymbol{\omega} \times \mathbf{v} \\ W_{kj} v_j \mathbf{e}_k &= \omega_i v_j \mathbf{e}_k \varepsilon_{ijk} \\ (W_{kj} v_j \mathbf{e}_k) \cdot \mathbf{e}_l &= (\omega_i v_j \mathbf{e}_k \varepsilon_{ijk}) \cdot \mathbf{e}_l \\ W_{lj} v_j &= \omega_i v_j \varepsilon_{ijl} \\ W_{lj} v_j - \omega_i v_j \varepsilon_{ijl} &= 0 \\ (W_{lj} - \omega_i \varepsilon_{ijl})\, v_j &= 0 \, . \end{aligned} \tag{71}$$

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Since this must again hold for any v_i $(i = 1, 2, 3)$, we have,

$$W_{kj} = \omega_i \varepsilon_{ijk} \ . \tag{72}$$

Multiply both sides by ε_{kjl} and sum over k, j,

$$\begin{aligned} W_{kj}\varepsilon_{kjl} &= \omega_i \varepsilon_{ijk}\varepsilon_{kjl} \\ W_{kj}\varepsilon_{kjl} &= -\omega_i \varepsilon_{kji}\varepsilon_{kjl} \\ W_{kj}\varepsilon_{kjl} &= -\omega_i 2\delta_{il} \\ W_{kj}\varepsilon_{kjl} &= -2\omega_l \ . \end{aligned} \tag{73}$$

Thus we have,

$$\omega_l = -\frac{1}{2} W_{kj}\varepsilon_{kjl} \ . \tag{74}$$

3.2 Differentiation and integration

Problem:

Show that the following relationship holds.

$$\int_{\partial\Omega} \mathbf{u} \otimes \mathbf{n} dA = \int_{\Omega} \text{grad}(\mathbf{u}) dV \tag{75}$$

Solution:

$$\begin{aligned}\left[\int_{\partial\Omega} \mathbf{u} \otimes \mathbf{n}\right]_{ij} dA &= \int_{\partial\Omega} u_i n_j dA \\ &= \int_{\Omega} u_{i,j} dV \\ &= \left[\int_{\Omega} \text{grad}(\mathbf{u}) dV\right]_{ij}\end{aligned} \tag{76}$$

Problem:

Find the derivative of the scalar valued tensor argument function,

$$f(\mathbf{A}) = \text{tr}(\mathbf{A}^2) \tag{77}$$

Solution:

Taking the directional derivative,

$$\begin{aligned}&\frac{\partial \text{tr}(\mathbf{A}^2)}{\partial \mathbf{A}} \cdot \mathbf{H} \\ &= \left.\frac{d}{d\alpha}\right|_{\alpha=0} \text{tr}((\mathbf{A} + \alpha\mathbf{H})^2) \\ &= \left.\frac{d}{d\alpha}\right|_{\alpha=0} \text{tr}(\mathbf{A}^2 + \alpha\mathbf{H}\mathbf{A} + \alpha\mathbf{A}\mathbf{H} + \alpha^2\mathbf{H}^2) \\ &= \left.\frac{d}{d\alpha}\right|_{\alpha=0} \text{tr}(\mathbf{A}^2) + \alpha\text{tr}(\mathbf{A}\mathbf{H})\alpha\text{tr}(\mathbf{H}\mathbf{A}) + \text{tr}(\mathbf{H}^2) \\ &= \text{tr}(\mathbf{A}\mathbf{H}) + \text{tr}(\mathbf{A}\mathbf{H}) \\ &= 2\mathbf{1} : \mathbf{A}\mathbf{H} \\ &= 2\mathbf{A}^T : \mathbf{H}\end{aligned} \tag{78}$$

Thus we have that,

$$\frac{\partial \text{tr}(\mathbf{A}^2)}{\partial \mathbf{A}} = 2\mathbf{A}^T \tag{79}$$

Alternatively, we can compute this derivative through the simplified index notation.

$$\begin{aligned}\left[\frac{\partial \text{tr}(\mathbf{A}^2)}{\partial \mathbf{A}}\right]_{kl} &= \frac{\partial A_{ij} A_{ji}}{\partial A_{kl}} \\ &= \delta_{ik}\delta_{jl}A_{ji} + A_{ij}\delta_{jk}\delta_{il} \\ &= 2A_{lk} \\ &= 2\left(A^T\right)_{kl}\end{aligned} \tag{80}$$

Thus we again have the relation in equation (79).

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3.3 Rigid body motion

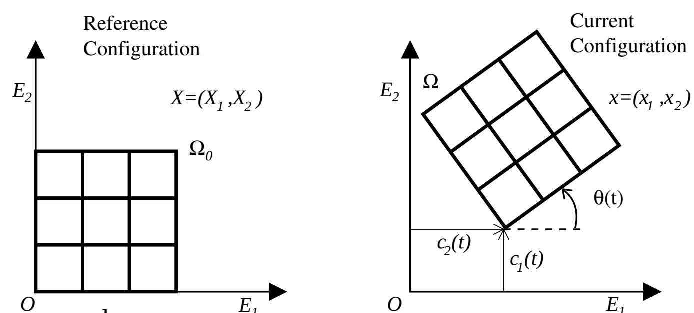
What is the motion for the deformation shown below. Assume that the problem is 2D, i.e. $\mathbf{X} = (X_1, X_2)$. The motion described is the rigid body rotation of a square around the origin O followed by a translation. Obtain the mapping,

$$\mathbf{x} = \varphi(\mathbf{X}, t)\ . \tag{81}$$

In other words express $\mathbf{x}$ in terms of $\mathbf{X}, t$. This mapping will have the form of,

$$\mathbf{x} = \mathbf{Q}(t)\mathbf{X} + \mathbf{c}(t) \tag{82}$$

where $\mathbf{Q}(t)$ will be an orthogonal tensor. Confirm this.



Solution:

The rotation $\mathbf{Q}(t)$ can be expressed as,

$$\mathbf{Q}(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \tag{83}$$

and thus the motion becomes,

$$\begin{aligned} \mathbf{x} &= \mathbf{Q}(t)\mathbf{X} + \mathbf{c}(t) \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} \end{aligned} \tag{84}$$