ETH Zurich
Department of Mechanical and Process Engineering
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Nonlinear Continuum Mechanics
Exercise 2

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Center of Mechanics

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1 Useful Definitions or Concepts

1.1 Notation

1.1.1 The representation of 2nd-order tensors and vectors as matrices and column vectors

When actual computation of 2nd-order tensors and vectors must be conducted, they must be represented in their component form. The components are given with respect to an orthonormal basis $\{e_i\}$. With respect to this basis, 2nd-order tensor $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ is represented as,

$$[\mathbf{A}]_{\mathbf{e}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}_{\mathbf{e}} \tag{1}$$

and vector $\mathbf{u} = u_i\mathbf{e}_i$ as,

$$[\mathbf{u}]_{\mathbf{e}} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_{\mathbf{e}} . \tag{2}$$

A bracket with a subscript $\{\mathbf{e}\}$ to denote the basis with which the components are taken is put around the 2nd-order tensor or vector to express the matrix or column vector form. With this rule of representation, one can confirm that,

$$\mathbf{v} = \mathbf{A}\mathbf{u} \tag{3}$$

can be represented as,

$$[\mathbf{v}]_{\mathbf{e}} = [\mathbf{A}]_{\mathbf{e}} [\mathbf{u}]_{\mathbf{e}} \tag{4}$$

and,

$$\mathbf{C} = \mathbf{A}\mathbf{B} \tag{5}$$

as,

$$[\mathbf{C}]_{\mathbf{e}} = [\mathbf{A}]_{\mathbf{e}} [\mathbf{B}]_{\mathbf{e}} . \tag{6}$$

Thus we see that standard operations of vectors and matrices hold in the form learned in linear algebra. The computation of the trace and determinant also follows the definitions given for matrices presented in linear algebra.

$$\text{tr}(\mathbf{A}) = \text{tr}([\mathbf{A}]_{\mathbf{e}}) \tag{7}$$

$$\det(\mathbf{A}) = \det([\mathbf{A}]_{\mathbf{e}}) \tag{8}$$

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1.1.2 Different representation of tensors

A vector or 2nd-order tensor can be represented in several different notations. Here we summarize three.

1. Symbolic notation(coordinate free): No dependence on coordinate system.

$$\mathbf{u} \qquad \mathbf{A} \qquad \mathbf{a}\otimes\mathbf{b}$$

2. Index notation: Representation with respect to a coordinate system $\{\mathbf{e}_i\}$.

$$u_i\mathbf{e}_i \quad A_{ij}\mathbf{e}_i\otimes\mathbf{e}_j \quad a_ib_j\mathbf{e}_i\otimes\mathbf{e}_j$$

3. Simplified index notation: A simplified version of the index notation where all tensors and vectors are expressed with respect to the same coordinate system $\{\mathbf{e}_i\}$. Thus we abreviate writing the portion corresponding to the basis. This can also be considered as looking at the components of the tensor or vector in their matrix representation.

$$u_i \qquad A_{ij} \qquad a_ib_j$$

It is important that one be accustomed to be able to move freely from one to another. Some simple examples are,

$$\begin{aligned}
\mathbf{c}=\mathbf{a}+\mathbf{b} &\Rightarrow & c_i\mathbf{e}_i = a_j\mathbf{e}_j + b_k\mathbf{e}_k &\Rightarrow & c_i = a_i+b_i\\
\mathbf{C}=\mathbf{A}+\mathbf{B} &\Rightarrow & C_{ij}\mathbf{e}_i\otimes\mathbf{e}_j = A_{kl}\mathbf{e}_k\otimes\mathbf{e}_l + B_{pq}\mathbf{e}_p\otimes\mathbf{e}_q &\Rightarrow & C_{ij} = A_{ij}+B_{ij}\\
\mathbf{v}=\mathbf{A}\mathbf{u} &\Rightarrow & v_i\mathbf{e}_i = A_{jk}u_k\mathbf{e}_j &\Rightarrow & v_i = A_{ij}u_j\\
\mathbf{C}=\mathbf{A}\mathbf{B} &\Rightarrow & C_{ij}\mathbf{e}_i\otimes\mathbf{e}_j = A_{ik}B_{kj}\mathbf{e}_i\otimes\mathbf{e}_j &\Rightarrow & C_{ij} = A_{ik}B_{kj}\ .
\end{aligned} \tag{9}$$

The convention that double indices are **always** summed still hold.

1.1.3 Free indices and dummy indices

When using indicial notation one must be take care whether the index is a free or dummy index.

- free: An index that is free from not being summed over.
- dummy: An index that is summed over, and can be replaced by any other symbol, i.e. a dummy.

In the expression,

$$v_i \quad = \quad A_{ij}u_j = A_{ik}u_k \tag{10}$$

j,k are dummy indices since they are summed over, and i is a free index. In an equation, the free indices must match between corresponding components. In the case above, we cannot have,

$$v_p \quad = \quad A_{ij}u_j \tag{11}$$

1.2 4th-order tensors

A 4th-order tensor $\mathcal{A}$ can be represented in index notation as,

$$A_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \tag{12}$$

or in simplified index notation as just,

$$A_{ijkl} \; . \tag{13}$$

The tensor product of two 2nd-order tensors $\mathbf{A}, \mathbf{B}$ also produces a 4th-order tensor,

$$\begin{aligned} \mathbf{A} \otimes \mathbf{B} &= A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j B_{kl} \otimes \mathbf{e}_k \otimes \mathbf{e}_l \\ &= A_{ij}B_{kl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \end{aligned} \tag{14}$$

or in simplified index notation as just,

$$A_{ij}B_{kl} \; . \tag{15}$$

The tensor product of four vectors can also produce a 4th-order tensor,

$$\begin{aligned} \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d} &= a_i\mathbf{e}_i \otimes b_j\mathbf{e}_j \otimes c_k\mathbf{e}_k \otimes d_l\mathbf{e}_l \\ &= a_i b_j c_k d_l \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \end{aligned} \tag{16}$$

or in simplified index notation as just,

$$a_i b_j c_k d_l \; . \tag{17}$$

The operations are defined as,

$$\begin{aligned} (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}) : (\mathbf{u} \otimes \mathbf{v}) &= [(\mathbf{c} \otimes \mathbf{d}) : (\mathbf{u} \otimes \mathbf{v})]\,(\mathbf{a} \otimes \mathbf{b}) \\ &= (\mathbf{c} \cdot \mathbf{u})(\mathbf{d} \cdot \mathbf{v})\mathbf{a} \otimes \mathbf{b} \end{aligned} \tag{18}$$

$$\begin{aligned} (\mathbf{w} \otimes \mathbf{z}) : (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}) &= [(\mathbf{w} \otimes \mathbf{z}) : (\mathbf{a} \otimes \mathbf{b})]\,(\mathbf{c} \otimes \mathbf{d}) \\ &= (\mathbf{w} \cdot \mathbf{a})(\mathbf{z} \cdot \mathbf{b})\mathbf{c} \otimes \mathbf{d} \end{aligned} \tag{19}$$

1.3 The triple scalar product

The triple scalar product of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is defined as,

$$[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}] = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \tag{20}$$

2 The permutation

The permutation ε_{ijk} is defined as follows,

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{(i,j,k)=(123),(231),(312)} \\ -1 & \text{(i,j,k)=(132),(213),(321)} \\ 0 & \text{(all other combinations)} \end{cases} . \tag{21}$$

Such that when the indices are in increasing order the value is postive 1, and when the indices are in decreasing order the value is negative 1. Otherwise it is 0. This definition is motivated by the need for defining an easy way to write the cross product. Observe the following relation for an orthonormal basis(right handed),

$$\begin{array}{rcl} \mathbf{e}_1 \times \mathbf{e}_2 = & \mathbf{e}_3 & = \varepsilon_{123}\mathbf{e}_3 \\ \mathbf{e}_2 \times \mathbf{e}_3 = & \mathbf{e}_1 & = \varepsilon_{231}\mathbf{e}_1 \\ \mathbf{e}_3 \times \mathbf{e}_1 = & \mathbf{e}_2 & = \varepsilon_{312}\mathbf{e}_2 \\ \mathbf{e}_1 \times \mathbf{e}_3 = & -\mathbf{e}_2 & = \varepsilon_{132}\mathbf{e}_2 \\ \mathbf{e}_2 \times \mathbf{e}_1 = & -\mathbf{e}_3 & = \varepsilon_{213}\mathbf{e}_3 \\ \mathbf{e}_3 \times \mathbf{e}_2 = & -\mathbf{e}_1 & = \varepsilon_{312}\mathbf{e}_1 \end{array} \tag{22}$$

and,

$$\begin{array}{ll} \mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{0} & = \varepsilon_{11k}\mathbf{e}_k \\ \mathbf{e}_2 \times \mathbf{e}_2 = \mathbf{0} & = \varepsilon_{22k}\mathbf{e}_k \\ \mathbf{e}_3 \times \mathbf{e}_3 = \mathbf{0} & = \varepsilon_{33k}\mathbf{e}_k \end{array} \tag{23}$$

where the k here is arbitrary. If we write this in more compact notation we obtain,

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk}\mathbf{e}_k . \tag{24}$$

Given this notation, we can easily express the cross product between vectors $\mathbf{a}$ and $\mathbf{b}$ as,

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_i\mathbf{e}_i) \times (b_j\mathbf{e}_j) \\ &= a_i b_j \mathbf{e}_i \times \mathbf{e}_j \\ &= a_i b_j \varepsilon_{ijk}\mathbf{e}_k \end{aligned} \tag{25}$$

Remarks:

The permutation has the properties that,

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} \tag{26}$$

and,

$$\varepsilon_{ijk} = -\varepsilon_{jik} . \tag{27}$$

There is exists the following relation between the permutation and the Kronecker delta,

1.

$$\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \tag{28}$$

Here i, j, p, q are free indices and must match on both sides of the equation. k is a dummy index.

2.

$$\varepsilon_{ijk}\varepsilon_{pjk} = 2\delta_{ip} \tag{29}$$

Here i, p are free indices and must match on both sides of the equation. j, k are dummy indices.

3.

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6 \tag{30}$$

3 Application of concepts or definitions

Problem:

What is the simplified index notation for,

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \tag{31}$$

Solution:

Take an inner product of both sides with $\mathbf{e}_i$.

$$\begin{aligned} \mathbf{c} \cdot \mathbf{e}_i &= (\mathbf{a} + \mathbf{b}) \cdot \mathbf{e}_i \\ c_i &= a_i + b_i \end{aligned} \tag{32}$$

Problem:

What is the simplified index notation for,

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{33}$$

Solution:

Apply $\mathbf{e}_i$ from the left, and apply $\mathbf{e}_j$ from the right on both sides of the equation.

$$\begin{aligned} \mathbf{e}_i \cdot \mathbf{C}\mathbf{e}_j &= \mathbf{e}_i \cdot (\mathbf{A} + \mathbf{B})\, \mathbf{e}_j \\ C_{ij} &= A_{ij} + B_{ij} \end{aligned} \tag{34}$$

Problem:

What is the simplified index notation for,

$$\mathbf{v} = \mathbf{A}\mathbf{u} \tag{35}$$

Solution:

Apply $\mathbf{e}_i$ from the left on both sides of the equation.

$$\begin{aligned} \mathbf{e}_i \cdot \mathbf{v} &= \mathbf{e}_i \cdot \mathbf{A}\mathbf{u} \\ &= \mathbf{e}_i \cdot \mathbf{A}\,(u_j \mathbf{e}_j) \\ v_i &= \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j u_j \\ &= A_{ij} u_j \end{aligned} \tag{36}$$

Problem:

What is the simplified index notation for,

$$\mathbf{C} = \mathbf{A}\mathbf{B} \tag{37}$$

Solution:

$$\begin{aligned}\mathbf{C} &= \mathbf{AB}\\ &= A_{pq}\mathbf{e}_p\otimes\mathbf{e}_q B_{rs}\mathbf{e}_r\otimes\mathbf{e}_s\\ &= A_{pq}B_{rs}\delta_{qr}\mathbf{e}_p\otimes\mathbf{e}_s\\ &= A_{pr}B_{rs}\mathbf{e}_p\otimes\mathbf{e}_s\end{aligned}$$

Apply $\mathbf{e}_i$ from the left, and apply $\mathbf{e}_j$ from the right on both sides of the equation.

$$\begin{aligned}\mathbf{e}_i\cdot\mathbf{C}\mathbf{e}_j &= \mathbf{e}_i\cdot\left(A_{pr}B_{rs}\mathbf{e}_p\otimes\mathbf{e}_s\right)\mathbf{e}_j\\ C_{ij} &= A_{pr}B_{rs}\delta_{ip}\delta_{sj}\\ &= A_{ir}B_{rj} \qquad (38)\end{aligned}$$

Problem:

Show that,

$$\mathbf{v} = \mathbf{Au} \qquad (39)$$

can be represented in the orthonormal basis $\{\mathbf{e}_i\}$ by the relationship,

$$[\mathbf{v}]_{\mathbf{e}} = [\mathbf{A}]_{\mathbf{e}}[\mathbf{u}]_{\mathbf{e}} \qquad (40)$$

Solution:

From the previous exercise we see that with respect to the orthonormal basis $\{\mathbf{e}_i\}$, we have

$$v_i = A_{ij}u_j \qquad (41)$$

between the components. If we write this in matrix notation we have,

$$\begin{aligned}\begin{bmatrix} v_1\\ v_2\\ v_3\end{bmatrix}_{\mathbf{e}} &= \begin{bmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}_{\mathbf{e}}\begin{bmatrix} u_1\\ u_2\\ u_3\end{bmatrix}_{\mathbf{e}} \qquad (42)\\ [\mathbf{v}]_{\mathbf{e}} &= [\mathbf{A}]_{\mathbf{e}}[\mathbf{u}]_{\mathbf{e}} \qquad (43)\end{aligned}$$

Problem:

Show that,

$$\mathbf{C} = \mathbf{AB} \qquad (44)$$

can be represented in the orthonormal basis $\{\mathbf{e}_i\}$ by the relationship,

$$[\mathbf{C}]_{\mathbf{e}} = [\mathbf{A}]_{\mathbf{e}}[\mathbf{B}]_{\mathbf{e}} \qquad (45)$$

Solution:

From the previous exercise we see that with respect to the orthonormal basis $\{\mathbf{e}_i\}$, we have

$$C_{ij} = A_{ik}B_{kj} \qquad (46)$$

between the components. If we write this in matrix notation we have,

$$\begin{aligned}\begin{bmatrix} C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}_{\mathbf{e}} &= \begin{bmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}_{\mathbf{e}}\begin{bmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}_{\mathbf{e}} \qquad (47)\\ [\mathbf{C}]_{\mathbf{e}} &= [\mathbf{A}]_{\mathbf{e}}[\mathbf{B}]_{\mathbf{e}} \qquad (48)\end{aligned}$$

Problem:

What is the simplified index notation for,

$$\mathbf{w} = \mathbf{A}\mathbf{u} + \mathbf{v} \tag{49}$$

Solution:

Apply $\mathbf{e}_i$ from the left on both sides of the equation.

$$\begin{aligned} \mathbf{e}_i \cdot \mathbf{w} &= \mathbf{e}_i \cdot (\mathbf{A}\mathbf{u} + \mathbf{v}) \\ w_i &= \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j u_j + v_i \,. \end{aligned} \tag{50}$$

Here we have used the relationship shown previously in this handout.

Problem:

Confirm that,

$$\mathbf{a} \times \mathbf{b} = a_i b_j \varepsilon_{ijk} \mathbf{e}_k \tag{51}$$

and the expression that you know,

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{52}$$

$$= \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \tag{53}$$

are the same.

Problem:

Show that,

$$[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}] = a_i b_j c_k \varepsilon_{ijk} \tag{54}$$

$$= \det\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \right) \tag{55}$$

4 Exercise

Problem:

Show that,

$$\left(\mathbf{A}^T\right)_{ij} = A_{ji} \tag{56}$$

Solution:

Apply $\mathbf{e}_i$ from the left, and apply $\mathbf{e}_j$ from the right.

$$\begin{aligned}\mathbf{e}_i \cdot \mathbf{A}^T\mathbf{e}_j &= \mathbf{A}\mathbf{e}_i \cdot \mathbf{e}_j \\ &= \mathbf{e}_j \cdot \mathbf{A}\mathbf{e}_i \\ &= A_{ji}\end{aligned} \tag{57}$$

Problem:

What is the simplified index notation for,

$$\mathbf{C} = \mathbf{A}^T + \mathbf{B} \tag{58}$$

Solution:

Apply $\mathbf{e}_i$ from the left, and apply $\mathbf{e}_j$ from the right on both sides of the equation.

$$\begin{aligned}\mathbf{e}_i \cdot \mathbf{C}\mathbf{e}_j &= \mathbf{e}_i \cdot \left(\mathbf{A}^T + \mathbf{B}\right)\mathbf{e}_j \\ C_{ij} &= \left(A^T\right)_{ij} + B_{ij} \\ &= A_{ji} + B_{ij}\end{aligned} \tag{59}$$

Problem:

Show that,

$$(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) \tag{60}$$

Solution:

$$\begin{aligned}(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{c} \otimes \mathbf{d}) &= \text{tr}\left((\mathbf{a} \otimes \mathbf{b})^T (\mathbf{c} \otimes \mathbf{d})\right) \\ &= \text{tr}\left((\mathbf{b} \otimes \mathbf{a})(\mathbf{c} \otimes \mathbf{d})\right) \\ &= \text{tr}\left((\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \otimes \mathbf{d})\right) \\ &= \text{tr}\left((\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} \otimes \mathbf{d}\right) \\ &= (\mathbf{a} \cdot \mathbf{c})\,\text{tr}\,(\mathbf{b} \otimes \mathbf{d}) \\ &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})\end{aligned} \tag{61}$$

Problem:

Show that,

$$(\mathbf{w} \otimes \mathbf{z}) : (\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}) : (\mathbf{u} \otimes \mathbf{v}) = (\mathbf{a} \cdot \mathbf{w})(\mathbf{b} \cdot \mathbf{z})(\mathbf{c} \cdot \mathbf{u})(\mathbf{d} \cdot \mathbf{v}) \tag{62}$$

Solution:

Straightforward application of the definitions yield the following.

$$\begin{aligned}(\mathbf{w}\otimes\mathbf{z}):(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}\otimes\mathbf{d}):(\mathbf{u}\otimes\mathbf{v}) &= (\mathbf{w}\otimes\mathbf{z}):[(\mathbf{c}\otimes\mathbf{d}):(\mathbf{u}\otimes\mathbf{v})]\,(\mathbf{a}\otimes\mathbf{b}) \\ &= [(\mathbf{c}\otimes\mathbf{d}):(\mathbf{u}\otimes\mathbf{v})]\,[(\mathbf{w}\otimes\mathbf{z}):(\mathbf{a}\otimes\mathbf{b})] \\ &= (\mathbf{a}\cdot\mathbf{w})(\mathbf{b}\cdot\mathbf{z})(\mathbf{c}\cdot\mathbf{u})(\mathbf{d}\cdot\mathbf{v}) \qquad (63)\end{aligned}$$

5 Homework

5.1 Using the identity related with the permutation

Using the identity,

$$\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \tag{64}$$

show the relationship,

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\,\mathbf{u}\,. \tag{65}$$

5.2 The physical interpretation of the determinant

The determinant of a tensor $\mathbf{A}$ is related to the change of volume. In class a definition of the determinant was given as,

$$\det(\mathbf{A}) = \frac{[\mathbf{Aa}\ \ \mathbf{Ab}\ \ \mathbf{Ac}]}{[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}]} \tag{66}$$

for arbitrary vectors $\mathbf{a}, \mathbf{b}$. This can be rewritten as,

$$[\mathbf{Aa}\ \ \mathbf{Ab}\ \ \mathbf{Ac}] = \det(\mathbf{A})\,[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}]\,. \tag{67}$$

Since the triple scalar product can be interpreted as the volume of the parallelpiped 'spanned' by $\mathbf{a}, \mathbf{b}, \mathbf{c}$, this relationship says that the volume of the parallelpiped 'spanned' by the mapped vectors $\mathbf{Aa}$, $\mathbf{Ab}$, $\mathbf{Ac}$ is $\det(\mathbf{A})$ times this value.

Given vectors,

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathbf{e}}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathbf{e}}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathbf{e}} \tag{68}$$

and tensor

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{\mathbf{e}} \tag{69}$$

confirm that,

$$\text{Volume of parallelpiped spanned by a,b,c} = [\mathbf{a}\,\mathbf{b}\,\mathbf{c}] \tag{70}$$

$$\text{Volume of parallelpiped spanned by Aa,Ab,Ac} = [\mathbf{Aa}\,\mathbf{Ab}\,\mathbf{Ac}]\,. \tag{71}$$

Then compute $\det(\mathbf{A})$ and confirm that,

$$[\text{Volume of parallelpiped spanned by Aa,Ab,Ac}] = \det(\mathbf{A}) \times [\text{Volume of parallelpiped spanned by a,b,c}] \tag{72}$$

5.3 4th-order tensors

$$\begin{aligned}
\mathbf{1} \otimes \mathbf{1} &= (\delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) \otimes (\delta_{kl}\mathbf{e}_k \otimes \mathbf{e}_l) && (73)\\
&= \delta_{ij}\delta_{kl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l && (74)\\
\mathbb{I} &= \delta_{ik}\delta_{jl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l && (75)\\
\bar{\mathbb{I}} &= \delta_{il}\delta_{jk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l && (76)\\
\mathbb{I}_{\text{symm}} &= \frac{1}{2}\left(\mathbb{I} + \bar{\mathbb{I}}\right) && (77)\\
\mathbb{I}_{\text{skew}} &= \frac{1}{2}\left(\mathbb{I} - \bar{\mathbb{I}}\right) && (78)
\end{aligned}$$

Show the following,

$$\begin{aligned}
\mathbf{1} \otimes \mathbf{1} : \mathbf{A} &= \text{tr}\,(\mathbf{A})\,\mathbf{1} && (79)\\
\mathbb{I} : \mathbf{A} &= \mathbf{A} && (80)\\
\bar{\mathbb{I}} : \mathbf{A} &= \mathbf{A}^T && (81)\\
\mathbb{I}_{\text{symm}} : \mathbf{A} &= \mathbf{A}_{\text{symm}} && (82)\\
\mathbb{I}_{\text{skew}} : \mathbf{A} &= \mathbf{A}_{\text{skew}} && (83)
\end{aligned}$$

where $\mathbf{A}$ is an arbitrary 2nd-order tensor.