ETH Zurich
Department of Mechanical and Process Engineering
Winter 06/07
Nonlinear Continuum Mechanics
Exercise 10

Institute for Mechanical Systems
Center of Mechanics

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1 Useful Definitions or Concepts

1.1 Different forms of the mechanical boundary value problem

One can solve either of the three forms shown below to solve the mechanical boundary value problem. The weak form is the most general and has the least restrictions.

1.1.1 Strong form

Strong form: Let a body $\mathcal{B}$ (reference configuration) be given with loading from body forces $\mathbf{B}$ (force per unit mass) defined in $\mathcal{B}$ and surface tractions $\mathbf{T}$ defined on $\partial_T \mathcal{B} \subset \partial \mathcal{B}$, and fixed deformations $\bar{\boldsymbol{\varphi}}$ on $\partial_{\boldsymbol{\varphi}} \mathcal{B}$. Find a mapping $\boldsymbol{\varphi}$ that satisfies,

Equilibrium	$\text{Div}\,[\mathbf{P}(\boldsymbol{\varphi})] + \rho_0 \mathbf{B} = 0$	in $\mathcal{B}$	
Displacement boundary condition	$\boldsymbol{\varphi} = \bar{\boldsymbol{\varphi}}$	on $\partial_{\boldsymbol{\varphi}} \mathcal{B}$	(1)
Force boundary condition	$\mathbf{P}(\boldsymbol{\varphi})\mathbf{N} = \bar{\mathbf{T}}$	on $\partial_{\mathbf{T}} \mathcal{B}$	

There are not many problems that can be solved by hand in this form. The reason why this form of the mechanical boundary problem is called the strong form is because there is a stronger differentiablity requirement on quantaties such as the stress $\mathbf{P}$ in this form. For this form to hold, the 1st Piola-Kirchhoff stress must be differentiable.

1.1.2 Weak form

Weak form (Principle of virtual work): Let a body $\mathcal{B}$ (reference configuration) be given with loading from body forces $\mathbf{B}$ (force per unit mass) defined in $\mathcal{B}$ and surface tractions $\mathbf{T}$ defined on $\partial_T \mathcal{B} \subset \partial \mathcal{B}$, and fixed deformations $\bar{\boldsymbol{\varphi}}$ on $\partial_{\boldsymbol{\varphi}} \mathcal{B}$. Find a mapping $\boldsymbol{\varphi}$ that satisfies,

$$\int_B \mathbf{P} : \text{Grad}[\delta\boldsymbol{\varphi}]\, d\mathbf{X} = \int_B \rho_0 \mathbf{B} \cdot \delta\boldsymbol{\varphi}\, d\mathbf{X} + \int_{\partial_T B} \bar{\mathbf{T}} \cdot \delta\boldsymbol{\varphi}\, dA \quad (\forall \delta\boldsymbol{\varphi} \text{ such that } \delta\boldsymbol{\varphi}(\mathbf{X}) = 0 \text{ on } \partial_{\boldsymbol{\varphi}}\mathbf{B}) \tag{2}$$

The $\delta\boldsymbol{\varphi}$ are called test functions and are called admissible when they satisfy the restriction stated above, that they must be zero on the portion of the boundary where the displacement is prescribed. As one can see, in this form $\mathbf{P}$ does not necessarily have to differentiable, and the requirements are weaker. (It is a stronger requirement to require a function to be differentiable).

1.1.3 Principle of minimum potential energy

Principle of potential minimum energy: Let a body $\mathcal{B}$ (reference configuration) be given with loading from body forces $\mathbf{B}$ (force per unit mass) defined in $\mathcal{B}$ and surface tractions $\mathbf{T}$ defined on $\partial_T \mathcal{B} \subset \partial \mathcal{B}$, and fixed deformations $\bar{\boldsymbol{\varphi}}$ on $\partial_{\boldsymbol{\varphi}} \mathcal{B}$. The body forces and surface tractions are assumed conservative, ie. indpendent of the deformation (dead loading). The material is assumed hyperelastic, such that $\mathbf{P} = \frac{\partial \Psi}{\partial \mathbf{F}}$. Find a mapping $\boldsymbol{\varphi}$ that satisfies $\boldsymbol{\varphi} = \overline{\boldsymbol{\varphi}}$ on $\partial_{\boldsymbol{\varphi}}\mathbf{B}$, and minimizes the potential energy $\Pi(\boldsymbol{\varphi}$ defined as,

$$\Pi(\boldsymbol{\varphi}) = \int_B \Psi(\boldsymbol{\varphi})\, d\mathbf{X} + \Pi_{\text{ext}}(\boldsymbol{\varphi}) \tag{3}$$

$$\Pi_{\text{ext}}(\boldsymbol{\varphi}) = -\int_B \rho_0 \mathbf{B} \cdot \boldsymbol{\varphi}\, d\mathbf{X} - \int_{\partial_T B} \bar{\mathbf{T}} \cdot \boldsymbol{\varphi}\, dA \tag{4}$$

The important assumptions here are,

1. The material is hyperelastic:

$$\mathbf{P} = \frac{\partial \Psi}{\partial \mathbf{F}} \tag{5}$$

2. Conservative loading: The surface force and body forces are independent of the deformation (dead loading). Examples of conservative are loads such as gravity. On the other hand, examples of non-conservative loading is pressure which is always perpendicular to the deforming surface, and follower loads.

1.1.4 Relationship between the formulations

The weak form is the most general and is related with the others in the following manner.

Strong form → Weak form ← Principle of minimum potential energy

The weak form implies the strong form when the stress $\mathbf{P}$ is differentiable. The weak form implies the Principle of minimum potential energy when the material is hyperelastic and the loading is conservative.

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2 Applications

Problem:

Show that the strong form implies the weak form.

Solution:

As in the previous exercise, take a dot product of the equilibirium equation with an admissible test function $\delta\boldsymbol{\varphi}$, and integrate over the domain $\mathcal{B}$, then conduct integration by parts and apply the force boundary conditions.

Problem:

Show that the principlal of minimum potential energy implies the weak form.

Solution:

Here we must show that the $\boldsymbol{\varphi}$ satisfying the boundary conditions and minimizing the potential energy, suffices the weak form. If $\boldsymbol{\varphi}$ makes the potential energy Π minimum, the energy should be stationary at this point, i.e. the slope should be zero or change of value should be zero to first order. If the function is perturbed in an arbitrary direction $\delta\boldsymbol{\varphi}$ by a magnitude of η,

$$\lim_{\eta\to 0} \frac{\Pi(\boldsymbol{\varphi}+\eta\delta\boldsymbol{\varphi}) - \Pi(\boldsymbol{\varphi})}{\eta} \quad = \quad 0\,. \tag{6}$$

Since $\boldsymbol{\varphi} + \eta\delta\boldsymbol{\varphi}$ must also satisfy the boundary conditions, $\delta\boldsymbol{\varphi}$ must be equal to zero on the portion of the boundary where the displacement is prescribed.

$$\delta\boldsymbol{\varphi} = 0 \quad \text{on } \partial_{\varphi}\mathcal{B} \tag{7}$$

The quantity in eqn. (6) is often called the first variation and is denoted $\delta\Pi$. So the requirement for $\boldsymbol{\varphi}$ to minimize Π is $\delta\Pi = 0$.

$$\begin{aligned}
0 &= \delta\Pi \\
&= \lim_{\eta\to 0} \frac{\Pi(\boldsymbol{\varphi}+\eta\delta\boldsymbol{\varphi}) - \Pi(\boldsymbol{\varphi})}{\eta} \\
&= \frac{d}{d\eta}\Big|_{\eta=0} \Pi(\boldsymbol{\varphi}+\eta\delta\boldsymbol{\varphi}) \\
&= \int_B \frac{d\Psi(\boldsymbol{\varphi}+\eta\delta\boldsymbol{\varphi})}{d\eta}\Big|_{\eta=0} d\mathbf{X} + \frac{d\Pi_{\text{ext}}}{d\eta}\Big|_{\eta=0} \\
&= \int_B \frac{\partial\Psi}{\partial\mathbf{F}} : \frac{d\mathbf{F}(\boldsymbol{\varphi}+\eta\delta\boldsymbol{\varphi})}{d\eta}\Big|_{\eta=0} d\mathbf{X} - \int_B \rho_0 \mathbf{B}\cdot\delta\boldsymbol{\varphi}\, d\mathbf{X} - \int_{\partial_T B} \bar{\mathbf{T}}\cdot\delta\boldsymbol{\varphi}\, dA \\
&= \int_B \frac{\partial\Psi}{\partial\mathbf{F}} : \text{Grad}\,[\delta\boldsymbol{\varphi}]\; d\mathbf{X} - \int_B \rho_0 \mathbf{B}\cdot\delta\boldsymbol{\varphi}\, d\mathbf{X} - \int_{\partial_T B} \bar{\mathbf{T}}\cdot\delta\boldsymbol{\varphi}\, dA
\end{aligned} \tag{8}$$

Since this holds for any admissible $\delta\boldsymbol{\varphi}$, this implies the weak form.

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Problem:

When the material is hyperelastic and the system has conservative loading, the mechanical problem can be solved through computing the deformation φ which minimized the potential energy. To be able to find a solution(existence) and to only have one solution(uniqueness), there are certain requirments on the stored energy function Ψ. In the linear elastic case, for the existence and uniqueness of solutions, convexity of the stored energy function was required. For the nonlinear case convexity is an unrealistic requirement. Why?

Solution:

In the nonlinear case, if the stored energy function Ψ is convex, there existence and uniqueness of solution can be proved. But this violates the physical aspects of the problem. The uniqueness of solution first is unrealistic since in the nonlinear problem there are multiple solutions.

- The nonlinear problem can have multiple solutions.

Some examples are shown below.

Convexity also violates the requirement called the growth condition. This condition requires the stored energy function to go to infinity when either the material is compressed to a point or when it is stretched infinitely. This relates coincides with the actual problem since it should take infinite amount of energy to do either. *F*or an example of this see the handout posted on the course website on convexity. Through frame indifference, convexity restricts the principle values of the Cauchy stress tensor to satisfy the relation,

$$\sigma_1 + \sigma_2 \geq 0, \;\; \sigma_2 + \sigma_3 \geq 0, \;\; \sigma_3 + \sigma_1 \geq 0 \, . \tag{9}$$

Problem:

Compute the geometric and material tangent stiffness from the weak form.

Solution:

Define,

$$\begin{aligned}
G(\boldsymbol{\varphi};\delta\boldsymbol{\varphi}) &= \int_B \frac{\partial\Psi}{\partial\mathbf{F}} : \text{Grad}\,[\delta\boldsymbol{\varphi}]\ d\mathbf{X} - \int_B \rho_0\mathbf{B}\cdot\delta\boldsymbol{\varphi}\, d\mathbf{X} - \int_{\partial_T B} \bar{\mathbf{T}}\cdot\delta\boldsymbol{\varphi}\, dA \\
&= \int_B \mathbf{P} : \text{Grad}\,[\delta\boldsymbol{\varphi}]\ d\mathbf{X} - \int_B \rho_0\mathbf{B}\cdot\delta\boldsymbol{\varphi}\, d\mathbf{X} - \int_{\partial_T B} \bar{\mathbf{T}}\cdot\delta\boldsymbol{\varphi}\, dA \\
&= \int_B \mathbf{F}\mathbf{S} : \text{Grad}\,[\delta\boldsymbol{\varphi}]\ d\mathbf{X} - \int_B \rho_0\mathbf{B}\cdot\delta\boldsymbol{\varphi}\, d\mathbf{X} - \int_{\partial_T B} \bar{\mathbf{T}}\cdot\delta\boldsymbol{\varphi}\, dA \\
&= \int_B \mathbf{S} : \mathbf{F}^T\text{Grad}\,[\delta\boldsymbol{\varphi}]\ d\mathbf{X} - \int_B \rho_0\mathbf{B}\cdot\delta\boldsymbol{\varphi}\, d\mathbf{X} - \int_{\partial_T B} \bar{\mathbf{T}}\cdot\delta\boldsymbol{\varphi}\, dA \qquad (10)\\
&= \int_B \mathbf{S} : \mathbf{F}^T\delta\mathbf{F}\, d\mathbf{X} - \int_B \rho_0\mathbf{B}\cdot\delta\boldsymbol{\varphi}\, d\mathbf{X} - \int_{\partial_T B} \bar{\mathbf{T}}\cdot\delta\boldsymbol{\varphi}\, dA \qquad (11)\\
\delta\boldsymbol{\varphi} &= \text{Grad}\,[\delta\boldsymbol{\varphi}] \qquad (12)\\
&= \text{grad}\,[\delta\boldsymbol{\varphi}]\,\mathbf{F} \qquad (13)\\
\Delta\boldsymbol{\varphi} &= \text{Grad}\,[\Delta\boldsymbol{\varphi}] \qquad (14)\\
&= \text{grad}\,[\Delta\boldsymbol{\varphi}]\,\mathbf{F} \qquad (15)\\
\delta\mathbf{C} &= 2\text{sym}[\mathbf{F}^T\delta\mathbf{F}] \qquad (16)\\
\Delta\mathbf{C} &= 2\text{sym}[\mathbf{F}^T\Delta\mathbf{F}] \qquad (17)\\
\mathbf{F}^\eta &= \text{Grad}\,[\boldsymbol{\varphi}+\eta\Delta\boldsymbol{\varphi}] \qquad (18)\\
\mathbf{C}^\eta &= \mathbf{F}^{\eta,T}\mathbf{F}^\eta \qquad (19)\\
\mathbb{C} &= 4\frac{\partial^2\Psi}{\partial\mathbf{C}^2} \qquad (20)\\
c_{ijkl} &= \frac{1}{J}F_{iA}F_{jB}F_{kC}F_{lD}\mathbb{C}_{ABCD}\ . \qquad (21)
\end{aligned}$$

Additionally compute the following quantities.

$$\begin{aligned}
\left.\frac{d\mathbf{C}^\eta}{d\eta}\right|_{\eta=0} &= \left.\Delta\mathbf{F}^T\mathbf{F}^\eta + \mathbf{F}^{\eta,T}\Delta\mathbf{F}\right|_{\eta=0} \\
&= \Delta\mathbf{F}^T\mathbf{F} + \mathbf{F}^T\Delta\mathbf{F} \\
&= 2\text{sym}[\mathbf{F}^T\Delta\mathbf{F}] \\
&= \Delta\mathbf{C} \qquad (22)\\
\frac{\partial\mathbf{S}}{\partial\mathbf{C}} &= \frac{\partial}{\partial\mathbf{C}}2\frac{\partial\Psi}{\partial\mathbf{C}} \\
&= 2\frac{\partial^2\Psi}{\partial\mathbf{C}^2} \qquad (23)\\
&= \frac{1}{2}\mathbb{C} \qquad (24)\\
& \qquad (25)
\end{aligned}$$

Take a direction derivative of G at φ in the direction of $\Delta\varphi$.(The body and traction forces do not depend on φ).

$$
\begin{aligned}
DG(\boldsymbol{\varphi};\delta\boldsymbol{\varphi})[\Delta\boldsymbol{\varphi}] &= \frac{d}{d\eta}\Big|_{\eta=0} G(\boldsymbol{\varphi}+\eta\Delta\boldsymbol{\varphi};\delta\boldsymbol{\varphi}) \\
&= \frac{d}{d\eta}\Big|_{\eta=0}\int_B \mathbf{S}^{\eta} : \mathbf{F}^{\eta,T}\delta\mathbf{F}\; d\mathbf{X} \\
&= \int_B \frac{d\mathbf{S}^{\eta}}{d\eta} : \mathbf{F}^{\eta,T}\delta\mathbf{F} + \mathbf{S}^{\eta} : \frac{d\mathbf{F}^{\eta,T}}{d\eta\delta\mathbf{F}\; d\mathbf{X}}\Big|_{\eta=0} \\
&= \int_B \left(\frac{\partial\mathbf{S}}{\partial\mathbf{C}} : \frac{d\mathbf{C}^{\eta}}{d\eta}\right) : \mathbf{F}^{\eta,T}\delta\mathbf{F} + \mathbf{S}^{\eta} : \Delta\mathbf{F}^{T}\delta\mathbf{F}\; d\mathbf{X}\Big|_{\eta=0} \\
&= \int_B \left(\frac{1}{2}\mathbb{C} : \frac{d\mathbf{C}^{\eta}}{d\eta}\right) : \mathbf{F}^{\eta,T}\delta\mathbf{F} + \mathbf{S}^{\eta} : \Delta\mathbf{F}^{T}\delta\mathbf{F}\; d\mathbf{X}\Big|_{\eta=0} \\
&= \int_B \left(\frac{1}{2}\mathbb{C} : \Delta\mathbf{C}\right) : \mathbf{F}^{T}\delta\mathbf{F} + \mathbf{S} : \Delta\mathbf{F}^{T}\delta\mathbf{F}\; d\mathbf{X} \\
&= \int_B \left(\frac{1}{2}\mathbb{C} : \Delta\mathbf{C}\right) : \mathrm{sym}[\mathbf{F}^{T}\delta\mathbf{F}] + \mathbf{S} : \Delta\mathbf{F}^{T}\delta\mathbf{F}\; d\mathbf{X} \\
&= \int_B \left(\frac{1}{4}\mathbb{C} : \Delta\mathbf{C}\right) : \delta\mathbf{C} + \mathbf{S} : \Delta\mathbf{F}^{T}\delta\mathbf{F}\; d\mathbf{X} \\
&= \int_B \Delta\mathbf{C} : \frac{1}{4}\mathbb{C} : \delta\mathbf{C} + \mathbf{S} : \Delta\mathbf{F}^{T}\delta\mathbf{F}\; d\mathbf{X} && (26) \\
&= \int_B \mathrm{sym}[\mathbf{F}^{T}\Delta\mathbf{F}] : \mathcal{C} : \mathrm{sym}[\mathbf{F}^{T}\delta\mathbf{F}] + \mathbf{S} : \Delta\mathbf{F}^{T}\delta\mathbf{F}\; d\mathbf{X} \\
&= \int_B \mathbf{F}^{T}\Delta\mathbf{F} : \mathbb{C} : \mathbf{F}^{T}\delta\mathbf{F} + \mathbf{S} : \Delta\mathbf{F}^{T}\delta\mathbf{F}\; d\mathbf{X} \\
&= \int_B F_{iA}\Delta F_{iB}\mathbb{C}_{ABCD}F_{kC}\delta F_{kD} + \mathbf{S} : \Delta\mathbf{F}^{T}\delta\mathbf{F}\; d\mathbf{X} \\
&= \int_B \Delta F_{iB}F_{iA}\mathbb{C}_{ABCD}F_{kC}\delta F_{kD} + \mathbf{S} : \Delta\mathbf{F}^{T}\delta\mathbf{F}\; d\mathbf{X} && (27) \\
&= \int_B \mathrm{grad}\,[\Delta\boldsymbol{\varphi}]_{i,j}\, F_{jB}F_{iA}\mathbb{C}_{ABCD}F_{kC}\mathrm{grad}\,[\delta\boldsymbol{\varphi}]_{k,l}\, F_{lD} + \mathbf{S} : (\mathrm{grad}\,[\Delta\boldsymbol{\varphi}]\,\mathbf{F})^{T}\mathrm{grad}\,[\Delta\boldsymbol{\varphi}]\,\mathbf{F}\; d\mathbf{X} && (28) \\
&= \int_B \mathrm{grad}\,[\Delta\boldsymbol{\varphi}]_{i,j}\, F_{iA}F_{jB}F_{kC}F_{lD}\mathbb{C}_{ABCD}\mathrm{grad}\,[\delta\boldsymbol{\varphi}]_{k,l} + \mathbf{F}\mathbf{S}\mathbf{F}^{T} : \mathrm{grad}\,[\Delta\boldsymbol{\varphi}]^{T}\,\mathrm{grad}\,[\Delta\boldsymbol{\varphi}]\; d\mathbf{X} && (29) \\
&= \int_S \mathrm{grad}\,[\Delta\boldsymbol{\varphi}]_{i,j}\,\frac{1}{J}F_{iA}F_{jB}F_{kC}F_{lD}\mathbb{C}_{ABCD}\mathrm{grad}\,[\delta\boldsymbol{\varphi}]_{k,l} + \frac{1}{J}\mathbf{F}\mathbf{S}\mathbf{F}^{T} : \mathrm{grad}\,[\Delta\boldsymbol{\varphi}]^{T}\,\mathrm{grad}\,[\Delta\boldsymbol{\varphi}]\; d\mathbf{x} && (30) \\
&= \int_S \mathrm{grad}\,[\delta\boldsymbol{\varphi}]_{i,j}\, c_{ijkl}\mathrm{grad}\,[\delta\boldsymbol{\varphi}]_{k,l} + \boldsymbol{\sigma} : \mathrm{grad}\,[\Delta\boldsymbol{\varphi}]^{T}\,\mathrm{grad}\,[\Delta\boldsymbol{\varphi}]\; d\mathbf{x} && (31)
\end{aligned}
$$

The first part is the material stiffness and the second is the geometric stiffness.

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Problem:

Derive the partial differential equation for beam buckling. A schematic of the beam is shown in Figure 1 From the strong form, formulate the weak form and potential energy for the beam pinned at both ends. (The axial force distribution is assumed to not depend on the deformation, i.e. conservative).

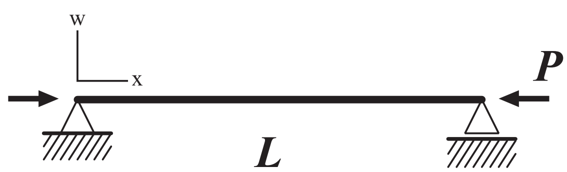

Figure 1: Beam schematic

Solution:

The moment balance for a infinitesimal piece of the beam is,

$$\begin{aligned} dM - Nw'dx - Qdx &= 0 \\ \frac{dM}{dx} - Nw' &= Q \end{aligned} \tag{32}$$

and the force balance is,

$$\begin{aligned} qdx - dQ &= 0 \\ \frac{dQ}{dx} &= q\,. \end{aligned} \tag{33}$$

The diagram of the force balance is shown in Figure 2.

Assuming the relationship between moment and curvature,

$$M = EI\frac{d^2w}{dx^2} \tag{34}$$

the beam equation is given as,

$$q = \frac{d^2}{dx^2}\left(EIw''\right) - \frac{d}{dx}\left(Nw'\right)\,. \tag{35}$$

In the following assume $q = 0$. To obtain the weak from, multiply the equilibrium equation by an admissible test function δw. Again for this function to be admissible it must be zero at the boundary where the displacement is prescribed. In this case,

$$\delta w(0) = 0 \tag{36}$$

$$\delta w(L) = 0\,. \tag{37}$$

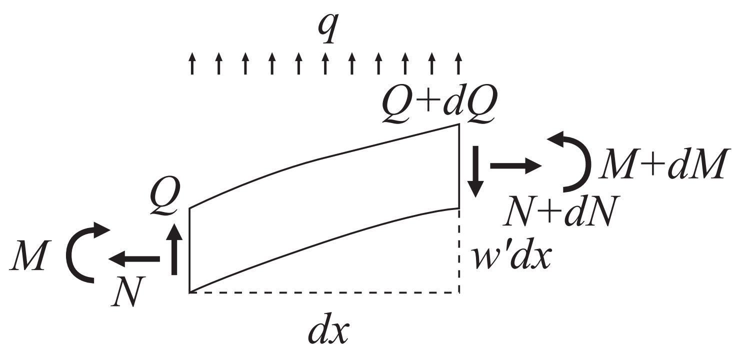

Figure 2: Force balance

The weak form is the following.

$$\begin{aligned} 0 &= \int_0^L \left(\frac{d^2}{dx^2} \left(EIw'' \right) - \frac{d}{dx} \left(Nw' \right) \right) \cdot \delta w dx \\ &= \int_0^L \frac{d^2}{dx^2} \left(EIw'' \right) \delta w - \frac{d}{dx} \left(Nw' \right) \delta w dx \\ &= \int_0^L - \frac{d}{dx} \left(EIw'' \right) \delta w' + \left(Nw' \right) \delta w' dx + \left[\frac{d}{dx} \left(EIw'' \right) \delta w - Nw' \delta w \right]_0^L \\ &= \int_0^L EIw'' \delta w'' + \left(Nw' \right) \delta w' dx + \left[\frac{d}{dx} \left(EIw'' \right) \delta w - EIw'' \delta' - Nw' \delta w \right]_0^L \\ &= \int_0^L EIw'' \delta w'' + \left(Nw' \right) \delta w' dx + \left[Q \delta w - M \delta w' \right]_0^L \ . \end{aligned} \tag{38}$$

Since the ends are pinned $M(0) = 0, M(L) = 0$, and since δw is admissible $\delta w(0) = 0, \delta w(L) = 0$. Thus the weak form becomes,

$$0 = \int_0^L EIw'' \delta w'' + \left(Nw' \right) \delta w' dx \ . \tag{39}$$

The first portion corresponds to the material stiffness, and the second portion to the geometric stiffness. If the axial force N is assumed independent of the the deformation and denoted $N = \overline{N}$, the principle of minimum potenntial energy can be utilized. The potential energy for this case is,

$$\Pi(w) = \int_0^L \frac{1}{2} EI (w'')^2 + \frac{1}{2} \overline{N} (w')^2 dx \ . \tag{40}$$

By taking the first variation, one can confirm that this implies the weak form.

$$
\begin{aligned}
\delta\Pi &= \frac{d}{d\eta}\Big|_{\eta=0}\Pi(w+\eta\delta w) \\
&= \frac{d}{d\eta}\Big|_{\eta=0}\int_0^L \frac{1}{2}EI([w+\eta\delta w]'')^2 + \frac{1}{2}\overline{N}([w+\eta\delta w]')^2 dx \\
&= \int_0^L \frac{1}{2}EI(\delta w'')(w+\eta\delta w) + \frac{1}{2}\overline{N}(\delta w')(w+\eta\delta w)dx\Big|_{\eta=0} \\
&= \int_0^L EIw''\delta w'' + (Nw')\,\delta w' dx \;. \qquad (41)
\end{aligned}
$$

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Problem:

Estimate the buckling load using the Ritz method.

Solution:

First we assume a form of the function,

$$\begin{aligned} w &= \Sigma_{A=1}^{n} N^A(x) d_A \\ &= [N^1, \ldots, N^n] \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \\ &= \mathbf{N}^T \mathbf{d} \end{aligned} \tag{42}$$

where N^A satisfy the displacement boundary conditions and d_A are generalized degrees of freedom. In this case, the displacement boundary conditions are zero, which allows us to pick the admissible function δw the same as w.

$$\begin{aligned} \delta w &= \Sigma_{A=1}^{n} N^A(x) \delta d_A \\ &= \mathbf{N}^T \delta \mathbf{d} \end{aligned} \tag{43}$$

Inserting this into the weak form of the buckling equations yields,

$$\delta \mathbf{d}^T \int_0^L EI \frac{d^2\mathbf{N}}{dx^2} \frac{d^2\mathbf{N}^T}{dx^2} dx \mathbf{d} + \delta \mathbf{d}^T \int_0^L \overline{N} \frac{d\mathbf{N}}{dx} \frac{d\mathbf{N}^T}{dx} dx \mathbf{d} = 0. \tag{44}$$

The first term is the material stiffness and the second term is the geometric stiffness. Assuming a compressive force that depends on a parameter P such that $N_P = -P\tilde{N}_P$,

$$\delta \mathbf{d}^T \int_0^L EI \frac{d^2\mathbf{N}}{dx^2} \frac{d^2\mathbf{N}^T}{dx^2} dx \mathbf{d} = P \delta \mathbf{d}^T \int_0^L \tilde{N}_P \frac{d\mathbf{N}}{dx} \frac{d\mathbf{N}^T}{dx} dx \mathbf{d} \tag{45}$$

and by defining the matrices,

$$\mathbf{K}_{\text{mat}} = \int_0^L EI \frac{d^2\mathbf{N}}{dx^2} \frac{d^2\mathbf{N}^T}{dx^2} dx \tag{46}$$

$$\mathbf{K}_{\text{geom}} = \int_0^L \tilde{N}_P \frac{d\mathbf{N}}{dx} \frac{d\mathbf{N}^T}{dx} dx \tag{47}$$

one obtains,

$$\delta \mathbf{d}^T \mathbf{K}_{\text{mat}} \mathbf{d} = P \delta \mathbf{d}^T \mathbf{K}_{\text{geom}} \mathbf{d} \, . \tag{48}$$

Since this must hold for any $\delta \mathbf{d}$, this yields the eigenvalue problem to solve for the buckling loads,

$$\mathbf{K}_{\text{mat}} \mathbf{d} = P \mathbf{K}_{\text{geom}} \mathbf{d} \, . \tag{49}$$

The critical load is the smallest of the eigenvalues,

$$P_{\text{crit}} = \min_i P_i \tag{50}$$

The flexibility of the method allows us to select as many functions as we want for our approximation. For simplicity we pick,

$$w = dx(L-x) \tag{51}$$
$$N_1 = x(L-x)\,. \tag{52}$$

With this equation the following quantaties are obtained.

$$\tilde{N}_P = 1 \tag{53}$$
$$\frac{d\mathbf{N}}{dx} = L-2x \tag{54}$$
$$\frac{d^2\mathbf{N}}{dx^2} = -2 \tag{55}$$
$$\begin{aligned}\mathbf{K}_{\text{mat}} &= \int_0^L EI\frac{d^2\mathbf{N}}{dx^2}\frac{d^2\mathbf{N}^T}{dx^2}dx \\ &= \int_0^L EI4dx \\ &= 4EIL\end{aligned} \tag{56}$$
$$\begin{aligned}\mathbf{K}_{\text{geom}} &= \int_0^L \tilde{N}_P\frac{d\mathbf{N}}{dx}\frac{d\mathbf{N}^T}{dx}dx \\ &= \int_0^L (L-2x)^2dx \\ &= \frac{L^3}{3}\end{aligned} \tag{57}$$

Thus,

$$\begin{aligned}\mathbf{K}_{\text{mat}}\mathbf{d} &= P\mathbf{K}_{\text{geom}}\mathbf{d} \\ 4EILd &= P\frac{L^3}{3}d \\ P &= \frac{12EI}{L^2}\end{aligned} \tag{58}$$

and the buckling load is obtained. The buckling mode is the approximating function

This overestimates the exact critical load which is $\frac{EI\pi^2}{L^2}$. This is due to the Ritz procedure which gives the exact load when the function is the exact form and overestimates otherwise. Thus if we insert the exact shape for the approximation,

$$w = d\sin\left(\frac{\pi x}{L}\right) \tag{59}$$

the exact load is obtained by the 1-by-1 matrices,

$$\mathbf{K}_{\text{mat}} = EI\left(\frac{\pi}{L}\right)^3\frac{\pi}{2} \tag{60}$$
$$\mathbf{K}_{\text{geom}} = \left(\frac{\pi}{L}\right)\frac{\pi}{2}\,. \tag{61}$$

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3 Homework

3.1 Problem: Small deformation elasticity

The strong form for the mechanical boundary value problem for small deformation elasticity is given as,

Strong form: Let a body Ω be given with loading from body forces $\mathbf{b}$ defined in Ω and surface tractions $\mathbf{t}$ defined on $\partial_t\Omega \subset \partial\Omega$, and fixed deformations $\bar{\mathbf{u}}$ on $\partial_{\mathbf{u}}\Omega$. Find a displacement field $\mathbf{u}$ that satisfies,

$$\begin{array}{lll} \text{Equilibrium} & \operatorname{div}[\boldsymbol{\sigma}(\mathbf{u})] + \mathbf{b} = 0 & \text{in } \Omega \\ \text{Displacement boundary condition} & \mathbf{u} = \bar{\mathbf{u}} & \text{on } \partial_{\mathbf{u}}\Omega \\ \text{Force boundary condition} & \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \bar{\mathbf{t}} & \text{on } \partial_{\mathbf{t}}\Omega \end{array} \tag{62}$$

3.1.1 Derive the weak form

Hint:

Multiply the equilibrium equation with an admissible test function δu, integrate by parts, and apply the force boundary conditions. In this case for δu to be admissible, it must satisfy,

$$\delta u = 0 \quad \text{on } \partial_{\mathbf{u}}\Omega\ . \tag{63}$$

Define,

$$\delta\boldsymbol{\varepsilon} \quad = \quad \frac{1}{2}\left(\text{Grad}\,[\delta\mathbf{u}] + \text{Grad}\,[\delta\mathbf{u}]^T\right)\ . \tag{64}$$

3.1.2 Derive the Principle of minimum potential energy

Under the assumption of a hyperelastic material, in this case for some scalar function W

$$\boldsymbol{\sigma} \quad = \quad \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \tag{65}$$

and conservative loading ($\mathbf{b}, \bar{\mathbf{t}}$ is independent of the displacment field $\mathbf{u}$), construct the potential energy of the system, and show that it implies the weak form.

Hint:

Define,

$$\Pi_{\text{ext}}(\mathbf{u}) \quad = \quad -\int_\Omega \mathbf{b}\cdot\mathbf{u}\,d\mathbf{x} - \int_{\partial_{\mathbf{t}}\Omega} \bar{\mathbf{t}}\cdot\mathbf{u}\,da\ . \tag{66}$$

3.2 Problem: Buckling of beams

Consider the column in Figure 3, with an applied distributed axial load $\overline{n}$(=const) along the height. Estimate the critical load using the Ritz method. The governing equations for this problem are,

$$\frac{d^2}{dx^2}\left(EIw''\right) - \frac{d}{dx}\left(Nw'\right) = 0 \tag{67}$$

$$\frac{dN}{dx} + \overline{n} = 0\,. \tag{68}$$

The corresponding weak form of the first equation is,

$$\int_\Omega EIw''\delta w'' + Nw'\delta w'\, d\mathbf{x} = 0\,. \tag{69}$$

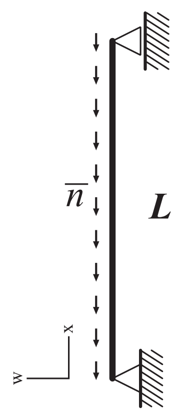

Figure 3: Beam schematic

3.2.1 Approximation 1.

Using the function,

$$w(x) = d\sin\left(\frac{\pi x}{L}\right) \tag{70}$$

as an approximation, compute an estimate of the critical load.

3.2.2 Approximation 2.

Using the function,

$$w(x) = d_1\sin\left(\frac{\pi x}{L}\right) + d_2\sin\left(\frac{3\pi x}{L}\right) \tag{71}$$

as an approximation, compute an estimate of the critical load. Comment on the results.

Hint:

Calculate $N(x)$ as a function of $\overline{n}$. Then use the weak form similar to the example and solve the scalar equation for $\overline{n}$ in the case of problem 1, and an eigenvalue problem in the case of problem 2.

Remark:

The exact solution involves solving a Bessel type of ODE leading to,

$$\overline{n}_{cr} = 18.5685 \frac{EI}{L^3} . \tag{72}$$