ETH Zurich
Department of Mechanical and Process Engineering
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Nonlinear Continuum Mechanics
Exercise 1

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1 Useful Definitions or Concepts

1.1 Extracting components from vectors or tensors

By taking the dot product of the vector or tensor with the basis of interest, the components can be extracted.

$$u_i = \mathbf{u} \cdot \mathbf{e}_i \quad (1)$$

$$A_{ij} = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j \quad (2)$$

This allows the following representation in components.

$$\mathbf{u} = u_i \mathbf{e}_i \quad (3)$$

$$\mathbf{A} = A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \quad (4)$$

1.2 Tensor products

The tensor product of two vectors $\mathbf{a}, \mathbf{b}$ is defined as,

$$\mathbf{a} \otimes \mathbf{b} \ . \quad (5)$$

The dot product with vectors are defined by the following. It must be noted that taking the dot product in front of and behind the tensor are different.

$$(\mathbf{a} \otimes \mathbf{b})\, \mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\, \mathbf{a} \quad (6)$$

$$\mathbf{w} \cdot (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{w} \cdot \mathbf{a})\, \mathbf{b} \quad (7)$$

In the Holzapfel book, Eqn. (7) is presented as,

$$\mathbf{w}\, (\mathbf{a} \otimes \mathbf{b}) = (\mathbf{w} \cdot \mathbf{a})\, \mathbf{b} \quad (8)$$

without the $(\cdot)$. We present the version with the dot for consistency but one must remember that this $\mathbf{a} \otimes \mathbf{b}$ is not a vector but a tensor and that it is not interchangable with $\mathbf{v}$.

$$\mathbf{w} \cdot (\mathbf{a} \otimes \mathbf{b}) \neq (\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{w} \quad (9)$$

1.3 Proving two thing are equal

In showing that two vectors $\mathbf{a}, \mathbf{b}$ are equal ($\mathbf{a} = \mathbf{b}$), this is equivalent to showing that the vector $\mathbf{a} - \mathbf{b}$ is equal to zero. This is equivalent to showing the following,

$$\forall \mathbf{v} \in \mathcal{V}, \quad (\mathbf{a} - \mathbf{b}) \cdot \mathbf{v} = \mathbf{0} \quad (10)$$

In the same manner, to show that two tensors $\mathbf{A}, \mathbf{B}$ are equal,

$$\forall \mathbf{v}, \mathbf{w} \in \mathcal{V}, \quad \mathbf{w} \cdot (\mathbf{A} - \mathbf{B})\, \mathbf{v} = \mathbf{0} \quad (11)$$

1.4 The trace operator and transpose

The trace of the tensor product is defined as,

$$\mathrm{tr}\, (\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} \quad (12)$$

and the transpose of a tensor is defined as,

$$\forall \mathbf{v}, \mathbf{w} \in \mathcal{V}, \quad \mathbf{w} \cdot \mathbf{A}^T \mathbf{v} = \mathbf{A}\mathbf{w} \cdot \mathbf{v} \quad (13)$$

2 Application of concepts or definitions

2.1 Show the component form of the dot product

Problem:

Show that,

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i \tag{14}$$

Solution:

Let $\mathbf{u} = u_i \mathbf{e}_i$ and $\mathbf{v} = v_j \mathbf{e}_j$. By inserting these in the relation above we obtain,

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (u_i \mathbf{e}_i) \cdot (v_j \mathbf{e}_j) \\ &= u_i v_j \, (\mathbf{e}_i \cdot \mathbf{e}_j) \\ &= u_i v_j \delta_{ij} \\ &= u_i v_i \; . \end{aligned} \tag{15}$$

2.2 Further examples of tensor products

Problem:

Prove the following relationship.

$$(\mathbf{a} \otimes \mathbf{b}) \cdot (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c}) \, \mathbf{a} \otimes \mathbf{d} \tag{16}$$

Solution:

Let $\mathbf{v}$ be an arbitrary vector. Apply this to the tensor as follows.

$$\begin{aligned} (\mathbf{a} \otimes \mathbf{b}) \cdot (\mathbf{c} \otimes \mathbf{d}) \, \mathbf{v} &= (\mathbf{a} \otimes \mathbf{b}) \, (\mathbf{d} \cdot \mathbf{v}) \, \mathbf{c} \\ &= (\mathbf{d} \cdot \mathbf{v}) \, (\mathbf{b} \cdot \mathbf{c}) \, \mathbf{a} \\ &= (\mathbf{b} \cdot \mathbf{c}) \, (\mathbf{d} \cdot \mathbf{v}) \, \mathbf{a} \\ &= (\mathbf{b} \cdot \mathbf{c}) \, (\mathbf{a} \otimes \mathbf{d}) \, \mathbf{v} \end{aligned} \tag{17}$$

Since this relationship holds for any vector $\mathbf{v}$, the two tensors are equal.

Problem:

Prove the relationship,

$$(\mathbf{a} \otimes \mathbf{b})^T = (\mathbf{b} \otimes \mathbf{a}) \; . \tag{18}$$

Solution:

Let $\mathbf{v}$ and $\mathbf{w}$ be arbitrary vectors. Apply this to the tensor as follows.

$$\begin{aligned} \mathbf{w} \cdot (\mathbf{a} \otimes \mathbf{b})^T \, \mathbf{v} &= (\mathbf{a} \otimes \mathbf{b}) \, \mathbf{w} \cdot \mathbf{v} \\ &= (\mathbf{b} \cdot \mathbf{w}) \, \mathbf{a} \cdot \mathbf{v} \\ &= \mathbf{w} \cdot (\mathbf{a} \cdot \mathbf{v}) \, \mathbf{b} \\ &= \mathbf{w} \cdot (\mathbf{b} \otimes \mathbf{a}) \, \mathbf{v} \end{aligned} \tag{19}$$

Since this relationship holds for any vectors $\mathbf{v}, \mathbf{w}$, the two tensors are equal.

Problem:

Prove the following relationship.

$$\mathbf{A}\left(\mathbf{u}\otimes\mathbf{v}\right) = \left(\mathbf{A}\mathbf{u}\right)\otimes\mathbf{v} \tag{20}$$

Solution:

Let $\mathbf{v}$ be an arbitrary vector. Apply this to the tensor as follows.

$$\begin{aligned}\mathbf{A}\left(\mathbf{a}\otimes\mathbf{b}\right)\mathbf{v} &= \mathbf{A}\left(\mathbf{b}\cdot\mathbf{v}\right)\mathbf{a} \\ &= \left(\mathbf{b}\cdot\mathbf{v}\right)\mathbf{A}\mathbf{a} \\ &= \left(\mathbf{A}\mathbf{a}\otimes\mathbf{b}\right)\mathbf{v}\end{aligned} \tag{21}$$

Since this relationship holds for any vector $\mathbf{v}$, the two tensors are equal.

3 Exercises

Problem:

Show that,

$$\mathbf{v} = \mathbf{A}\mathbf{u} \tag{22}$$

implies,

$$v_i = A_{ij} u_j \ . \tag{23}$$

Solution:

Let $\mathbf{u} = u_i \mathbf{e}_i$, $\mathbf{v} = v_j \mathbf{e}_j$, and $\mathbf{A} = A_{ij} \mathbf{e}_k \otimes \mathbf{e}_l$. Inserting this into the above expresion gives,

$$\begin{aligned} v_j \mathbf{e}_j &= A_{kl} \mathbf{e}_k \otimes \mathbf{e}_l u_i \mathbf{e}_i \\ &= A_{kl} u_i \left(\mathbf{e}_k \otimes \mathbf{e}_l \right) \mathbf{e}_i \\ &= A_{kl} u_i \delta_{li} \mathbf{e}_k \\ &= A_{ki} u_i \mathbf{e}_k \ . \end{aligned} \tag{24}$$

With this expression we take the inner product of both sides with basis vector $\mathbf{e}_p$,

$$\begin{aligned} v_j \mathbf{e}_j \cdot \mathbf{e}_p &= A_{ki} u_i \mathbf{e}_k \ \cdot \mathbf{e}_p \\ v_j \delta_{jp} &= A_{ki} u_i \delta_{kp} \\ v_p &= A_{pi} u_i \ . \end{aligned} \tag{25}$$

Thus we have the desired result.

Alternatively we could show the relation by the following process. Starting with the first expression we take the inner product with the basis vector $\mathbf{e}_p$.

$$\begin{aligned} \mathbf{e}_p \cdot \mathbf{v} &= \mathbf{e}_p \cdot \mathbf{A}\mathbf{u} \\ &= \mathbf{e}_p \cdot \mathbf{A} \left(u_i \mathbf{e}_i \right) \\ &= u_i \mathbf{e}_p \cdot \mathbf{A} \mathbf{e}_i \\ v_p &= u_i A_{pi} \end{aligned} \tag{26}$$

Problem:

Show that,

$$(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j \; . \tag{27}$$

Solution:

To obtain the component expression we select basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ and apply them to the tensor as follows.

$$\begin{aligned} \mathbf{e}_i \cdot (\mathbf{a} \otimes \mathbf{b})\, \mathbf{e}_j &= \mathbf{e}_i\, (\mathbf{b} \cdot \mathbf{e}_j)\, \mathbf{a} \\ &= b_j \mathbf{e}_i \cdot \mathbf{a} \\ &= a_i b_j \end{aligned} \tag{28}$$

Alternatively we can obtain the expression by the following.

$$\begin{aligned} \mathbf{a} \otimes \mathbf{b} &= (a_i \mathbf{e}_i) \otimes (b_j \mathbf{e}_j) \\ &= a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j \end{aligned} \tag{29}$$

Since we know that,

$$\mathbf{a} \otimes \mathbf{b} = (\mathbf{a} \otimes \mathbf{b})_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j \tag{30}$$

if we compare this with the expression above, we obtain the desired the result.

Problem:

Prove the following relation.

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \tag{31}$$

Solution:

Let $\mathbf{v}$ and $\mathbf{w}$ be arbitrary vectors. Apply this to the tensor as follows.

$$\begin{aligned} \mathbf{w}\cdot(\mathbf{AB})^T\mathbf{v} &= (\mathbf{AB})\,\mathbf{w}\cdot\mathbf{v} \\ &= \mathbf{Bw}\cdot\mathbf{A}^T\mathbf{v} \\ &= \mathbf{w}\cdot\mathbf{B}^T\mathbf{A}^T\mathbf{v} \end{aligned} \tag{32}$$

Since this relationship holds for any vectors $\mathbf{v}, \mathbf{w}$, the two tensors are equal.

Alternatively we can work with the components to see that the expressions are equal. Let $\mathbf{A} = A_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$ and $\mathbf{B} = B_{kl}\mathbf{e}_k\otimes\mathbf{e}_l$. Inserting this into the expression we have,

$$\begin{aligned} (\mathbf{AB})^T &= (A_{ij}\mathbf{e}_i\otimes\mathbf{e}_j\cdot B_{kl}\mathbf{e}_k\otimes\mathbf{e}_l)^T \\ &= (A_{ij}B_{kl}\mathbf{e}_i\otimes\mathbf{e}_j\cdot\mathbf{e}_k\otimes\mathbf{e}_l)^T \end{aligned}$$

By using Eqn. (16).

$$\begin{aligned} (\mathbf{AB})^T &= A_{ij}B_{kl}\,(\mathbf{e}_i\otimes\mathbf{e}_l\delta_{jk})^T \\ &= A_{ij}B_{kl}\delta_{jk}\,(\mathbf{e}_i\otimes\mathbf{e}_l)^T \end{aligned} \tag{33}$$

Using the relation shown in Eqn. (18), we have,

$$(\mathbf{AB})^T = A_{ij}B_{jl}\,(\mathbf{e}_l\otimes\mathbf{e}_i)\ . \tag{34}$$

Next we evaluate the right hand side of the problem.

$$\begin{aligned} \mathbf{B}^T\mathbf{A}^T &= (B_{kl}\mathbf{e}_k\otimes\mathbf{e}_l)^T\,(A_{ij}\mathbf{e}_i\otimes\mathbf{e}_j)^T \\ &= B_{kl}A_{ij}\,(\mathbf{e}_k\otimes\mathbf{e}_l)^T\,(\mathbf{e}_i\otimes\mathbf{e}_j)^T \\ &= B_{kl}A_{ij}\,(\mathbf{e}_l\otimes\mathbf{e}_k)\,(\mathbf{e}_j\otimes\mathbf{e}_i) \end{aligned} \tag{35}$$

Here we have again used Eqn. (18). Next we use Eqn. (16).

$$\begin{aligned} \mathbf{B}^T\mathbf{A}^T &= B_{kl}A_{ij}\delta_{kj}\mathbf{e}_l\otimes\mathbf{e}_i \\ &= B_{jl}A_{ij}\mathbf{e}_l\otimes\mathbf{e}_i \end{aligned} \tag{36}$$

Comparing Eqn. (33) and Eqn. (36) we obtain the desired result.

Problem:

Show the relationship,

$$\begin{aligned} \mathbf{A} : \mathbf{B} &= \text{tr}\left(\mathbf{A}^T \mathbf{B}\right) \\ &= A_{ij} B_{ij} \end{aligned} \tag{37}$$

Solution:

$$\begin{aligned} \mathbf{A} : \mathbf{B} &= \text{tr}\left(\mathbf{A}^T \mathbf{B}\right) \\ &= \text{tr}\left(\left(A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \right)^T \cdot B_{kl} \mathbf{e}_k \otimes \mathbf{e}_k \right) \\ &= A_{ij} B_{kl} \text{tr}\left(\mathbf{e}_j \otimes \mathbf{e}_i \cdot \mathbf{e}_k \otimes \mathbf{e}_l \right) \\ &= A_{ij} B_{kl} \text{tr}\left(\delta_{ik} \mathbf{e}_j \otimes \mathbf{e}_l \right) \\ &= A_{ij} B_{kl} \delta_{ik} \mathbf{e}_j \cdot \mathbf{e}_l \\ &= A_{ij} B_{il} \delta_{jl} \\ &= A_{ij} B_{ij} \end{aligned} \tag{38}$$

4 Homework

4.1 Show that the following relationships hold

$$\begin{aligned}
\operatorname{tr}(\mathbf{A}) &= \mathbf{1} : \mathbf{A} && (39)\\
\mathbf{A} : \mathbf{BC} &= \left(\mathbf{B}^T\mathbf{A}\right) : \mathbf{C} = \left(\mathbf{AC}^T\right) : \mathbf{B} && (40)\\
\mathbf{A} : (\mathbf{u} \otimes \mathbf{v}) &= \mathbf{u} \cdot \mathbf{Av} = (\mathbf{u} \otimes \mathbf{v}) : \mathbf{A} && (41)
\end{aligned}$$

Hint: Express the vectors and tensors in component form, i.e. $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ and insert in the relations above. Use the definition of the double contraction ":".

4.2 Projections operators

Given a direction $\mathbf{e}$, an arbitrary vector $\mathbf{a}$ can be decomposed into the component parallel to $\mathbf{e}$ which we denote $\mathbf{a}_\parallel$ and the component in the plane normal to $\mathbf{e}$ we denote as $\mathbf{a}_\perp$. They are defined as,

$$\begin{aligned}
\mathbf{a} &= \mathbf{a}_\parallel + \mathbf{a}_\perp && (42)\\
\mathbf{a}_\parallel &= (\mathbf{a} \cdot \mathbf{e})\,\mathbf{e} && (43)\\
\mathbf{a}_\perp &= \mathbf{a} - \mathbf{a}_\parallel && (44)
\end{aligned}$$

Corresponding to these we define projection tensors,

$$\begin{aligned}
\mathbf{P}_e^\parallel &= \mathbf{e} \otimes \mathbf{e} && (45)\\
\mathbf{P}_e^\perp &= \mathbf{1} - \mathbf{P}_e^\parallel && (46)
\end{aligned}$$

4.2.1 Verify that $\mathbf{P}_e^\parallel\mathbf{a} = \mathbf{a}_\parallel$, $\mathbf{P}_e^\perp\mathbf{a} = \mathbf{a}_\perp$, and $\mathbf{a}_\parallel \cdot \mathbf{a}_\perp = 0$

.

4.2.2 Verify the following expressions.

$$\begin{aligned}
\mathbf{P}_e^\parallel\mathbf{P}_e^\parallel &= \mathbf{P}_e^\parallel && (47)\\
\mathbf{P}_e^\perp\mathbf{P}_e^\perp &= \mathbf{P}_e^\perp && (48)\\
\mathbf{P}_e^\perp\mathbf{P}_e^\parallel &= \mathbf{0} && (49)
\end{aligned}$$

4.3 Decomposition into symmetric and skew parts

Every tensor $\mathbf{A}$ can be decomposed into its symmetric and skew parts. They are defined as follows.

$$\begin{aligned}
\mathbf{A} &= \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) + \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right)\\
&= \mathbf{A}_{\text{symm}} + \mathbf{A}_{\text{skew}} && (50)
\end{aligned}$$

where,

$$\begin{aligned}
\mathbf{A}_{\text{symm}} &= \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) && (51)\\
\mathbf{A}_{\text{skew}} &= \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right) && (52)
\end{aligned}$$

skew parts. Recall that a symmetric tensor $\mathbf{S}$ has the property,

$$\mathbf{S}^T = \mathbf{S} \tag{53}$$

and a skew tensor $\mathbf{W}$ has the property,

$$\mathbf{W}^T = -\mathbf{W} \tag{54}$$

4.3.1 Show that, $\mathbf{A}_{\text{symm}}$ is indeed symmetric and that $\mathbf{A}_{\text{skew}}$ is indeed skew. In other words show,

$$\mathbf{A}_{\text{symm}}^T = \mathbf{A}_{\text{symm}} \tag{55}$$

$$\mathbf{A}_{\text{skew}}^T = -\mathbf{A}_{\text{skew}} \tag{56}$$

4.3.2 Then show that for a symmetric tensor $\mathbf{S}$ and skew symmetric tensor $\mathbf{W}$,

$$\mathbf{S} : \mathbf{W} = 0 \tag{57}$$

Hint: Write out the expression in components and think about what structure the components of $\mathbf{S}$ and $\mathbf{W}$ have.