Uniqueness in linear elasticity

One important point to finding solutions in linear elasticity is that they are unique. This is a general feature of any type of linear problem with suitable boundary conditions. Thus if one finds *a solution* to a particular linear elastic boundary value problem then it is *the solution* to the problem. There are many ways to prove this and we shall review here the classical method based upon Clapeyron's Theorem. This theorem states:

Theorem 1 (Clapeyron's Theorem) *Consider a linear elastic body $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$. Then*

$$\int_{\partial\Omega} t_i u_i + \int_{\Omega} b_i u_i = 2 \int_{\Omega} W \,,$$

where t_i are the surface tractions, b_i is the body force, u_i is the displacement field, and W is the strain energy density.

The proof of the theorem is rather straightforward. Simply go through the derivation of the weak from using the displacement field as a "test function".

Now Consider that the boundary $\partial\Omega = \overline{\partial\Omega_u \bigcup \partial\Omega_t}$, where the traction and displacement parts of the boundary are mutually exclusive ($\partial\Omega_u \bigcap \partial\Omega_t = \emptyset$). On $\partial\Omega_t$, assume an imposed traction of $\bar{t}_i$. On $\partial\Omega_u$, assume an imposed displacement of $\bar{u}_i$. To show uniqueness of the solution let us assume that there are two different solutions to the problem and then show (by Clayperon's Theorem) that their difference must be zero.

Call the two solutions ($u_i^{(1)}$, $\varepsilon_{ij}^{(1)}$, $\sigma_{ij}^{(1)}$) and ($u_i^{(2)}$, $\varepsilon_{ij}^{(2)}$, $\sigma_{ij}^{(2)}$). Now define the difference between the two solutions as:

$$u_i^D = u_i^{(1)} - u_i^{(2)} \tag{1}$$

$$\varepsilon_{ij}^D = \varepsilon_{ij}^{(1)} - \varepsilon_{ij}^{(2)} \tag{2}$$

$$\sigma_{ij}^D = \sigma_{ij}^{(1)} - \sigma_{ij}^{(2)} \,. \tag{3}$$

Note that the difference functions are the solution to a problem with applied body force $b_i = 0$, and applied boundary conditions $\bar{t}_i = 0$ on $\partial\Omega_t$ and $\bar{u}_i = 0$

on $\partial\Omega_u$. If we apply Clapeyron's Theorem to this difference problem then we have that:

$$0 = \int_\Omega \varepsilon^D_{ij} \mathbb{C}_{ijkl} \varepsilon^D_{kl} \,. \tag{4}$$

If we assume that the integrand is a positive definite quadratic form, then we have that

$$\varepsilon^D_{ij} = 0 \,. \tag{5}$$

Note that this implies that $u^D_i = 0$ and that $\sigma^D_{ij} = 0$. Thus the difference between the two solutions is indeed zero and the solution is unique.

Remark: The assumption that $\mathbb{C}_{ijkl}$ be positive definite essentially says that we are assuming the material to be stable and it is perfectly compatible with our normal physical experience with stable materials.

Remark: The assumption that $\partial\Omega_u \bigcap \partial\Omega_t = \emptyset$ is essential for the existence of solutions. Note that this notation implies that at a given point in a given coordinate direction one can only impose a displacement or a traction.

Remark: Similar results hold in rate form for viscoelasticity and plasticity.