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The Symmetric Identity: $\mathbb{I}^{\mathrm{sym}} \to \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$

The derivative of a second order tensor with itself is the fourth order identity tensor $\mathbb{I}$, in the same way as the second order identity $\mathbf{1} = \partial \boldsymbol{x}/\partial \boldsymbol{x}$ when $\boldsymbol{x}$ is a vector. In components $\delta_{ij} = \partial x_i/\partial x_j$. For the case of interest, in components we have for a tensor $\boldsymbol{A}$ with components A_{ij}

$$\mathbb{I}_{ijkl} = \frac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik}\delta_{jl}\,; \tag{1}$$

i.e. if $i = k$ *and* $j = l$ then the result is unity and if one of the equalities is violated, then the result is zero. So, for example, $\partial A_{12}/\partial A_{12} = 1$ and $\partial A_{12}/\partial A_{21} = 0$ etc. This is as we expect if the components are independent.

When the tensor is symmetric, $\boldsymbol{A} \in \mathbb{S}$ (the set of symmetric tensors), then we come to the result that the identity is no longer given by (1); note the components of a symmetric second order tensor are not all independent. Rather it is given component-wise by the expression in the header. This then returns the somewhat non-intuitive result that $\partial A_{12}/\partial A_{12} = \mathbb{I}^{\mathrm{sym}}_{1212} = 1/2$ (not unity!) even though $\partial A_{11}/\partial A_{11} = \mathbb{I}^{\mathrm{sym}}_{1111} = 1$. How can this be?

Consider a scalar-valued function on symmetric tensors, $f : \mathbb{S} \to \mathbb{R}$. Its directional derivative[1] is a linear mapping $Df : \mathbb{S} \to \mathbb{R}$. As such we can represent it by a tensor, which we choose to call $\boldsymbol{B}$:

$$Df[\boldsymbol{C}] \equiv \boldsymbol{B} : \boldsymbol{C}\,. \tag{2}$$

The tensor $\boldsymbol{B}$ characterizing the derivative of $f(\cdot)$ in the (symmetric) direction $\boldsymbol{C}$ is non-unique since one can add an arbitrary skew-symmetric tensor to $\boldsymbol{B}$ without changing the result. The symmetric part of $\boldsymbol{B}$, viz. $\frac{1}{2}(\boldsymbol{B}+\boldsymbol{B}^T)$, is however unique. This is the tensor which we *choose* to use to represent the derivative Df. We do this for two reasons: (1) it is unique, thus there is no ambiguity, and (2) it will always produce the correct result for the rate of change of $f(\cdot)$ in a *symmetric* direction. Note it only makes sense to discuss rates of change of $f(\cdot)$ in symmetric directions, since $f(\cdot)$ is only defined over the space of symmetric tensors.

[1]Assuming it exists.

Let us now apply this result to the component extraction function $f(\boldsymbol{A}) = (\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : \boldsymbol{A} = A_{ij}$. To compute the derivative of $f(\cdot)$ we apply the directional derivative formula

$$\frac{d}{d\alpha}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (\boldsymbol{A} + \alpha \boldsymbol{H})\bigg|_{\alpha=0} = Df[\boldsymbol{H}]\,, \tag{3}$$

where $\boldsymbol{H} \in \mathbb{S}$. Taking the derivative with respect to α, setting α to zero, leads to:

$$(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : \boldsymbol{H} = Df[\boldsymbol{H}]\,, \tag{4}$$

which allows us to identify *an* expression for $\boldsymbol{B}$ as $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. The unique part of $\boldsymbol{B}$ which provides the derivative information is the symmetric part; viz. $\frac{1}{2}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j + \boldsymbol{e}_j \otimes \boldsymbol{e}_i)$. This expression gives us the derivative $\partial A_{ij}/\partial \boldsymbol{A}$. If we now compute its components we get

$$\frac{\partial A_{ij}}{\partial A_{kl}} = \frac{\partial A_{ij}}{\partial \boldsymbol{A}} : (\boldsymbol{e}_k \otimes \boldsymbol{e}_l) = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,. \tag{5}$$

Thus we arrive at the desired expression – an expression for the fourth order identity tensor over the space of symmetric tensors. Observe that this expression yields the results $\partial A_{11}/\partial A_{11} = \mathbb{I}^{\text{sym}}_{1111} = 1$, and the non-intuitive (but correct) results $\partial A_{12}/\partial A_{12} = \mathbb{I}^{\text{sym}}_{1212} = 1/2$, as well as $\partial A_{12}/\partial A_{21} = \mathbb{I}^{\text{sym}}_{1221} = 1/2$.