UNIVERSITY OF CALIFORNIA BERKELEY Structural Engineering,
Department of Civil Engineering Mechanics and Materials
Fall 2008 Professor: S. Govindjee

End effects in elastic systems

St. Venant's Principle gives the qualitative statement that the effect of self-equilibrated loads is confined to a region local to the application of the loads. Let us demonstrate the validity of this statement in a particular example. Consider a bar of width $2d$ and length L. Let assume that the loading is a uniform end load, σ_o, plus a self equilibrated contribution $\bar{\sigma}(y)$; i.e.

$$\int_{-d}^{d} \bar{\sigma}(y)\,dy = 0 \tag{1}$$

$$\int_{-d}^{d} y\bar{\sigma}(y)\,dy = 0\,. \tag{2}$$

The solution to this problem will be a uniform field $\boldsymbol{\sigma}(x,y) = \sigma_o \boldsymbol{e}_x \otimes \boldsymbol{e}_x$ plus a correction field, $\boldsymbol{\sigma}^c(x,y)$, which matches the boundary conditions:

$$\sigma^c_{xx}(0,y) = \bar{\sigma}(y) \tag{3}$$

$$\sigma^c_{xy}(0,y) = 0 \tag{4}$$

$$\sigma^c_{yy}(x,\pm d) = 0 \tag{5}$$

$$\sigma^c_{yx}(x,\pm d) = 0\,. \tag{6}$$

We will assume $L >> d$ and consequently will anticipate that $\boldsymbol{\sigma}^c \rightarrow \boldsymbol{0}$ as $x \rightarrow L$. Likewise, we anticipate a similar end-effect solution from the right. In keeping with the notion that the solution should drop off rapidly from the end of the bar let us assume an Airy stress function of the following form (for the correction stresses):

$$\phi = K\exp[-\gamma x/d]f(y)\,, \tag{7}$$

where K and γ are unknown constants and $f(y)$ is an unknown function. Our goal is to determine γ so that we can come to an understanding of how local the correction stresses really are. From this Ansatz, we have that:

$$\sigma_{xx} = \phi_{,yy} = K\exp[-\gamma x/d]f''(y) \tag{8}$$

$$\sigma_{yy} = \phi_{,xx} = K(\gamma/d)^2\exp[-\gamma x/d]f(y) \tag{9}$$

$$\sigma_{xy} = -\phi_{,xy} = K(\gamma/d)\exp[-\gamma x/d]f'(y)\,. \tag{10}$$

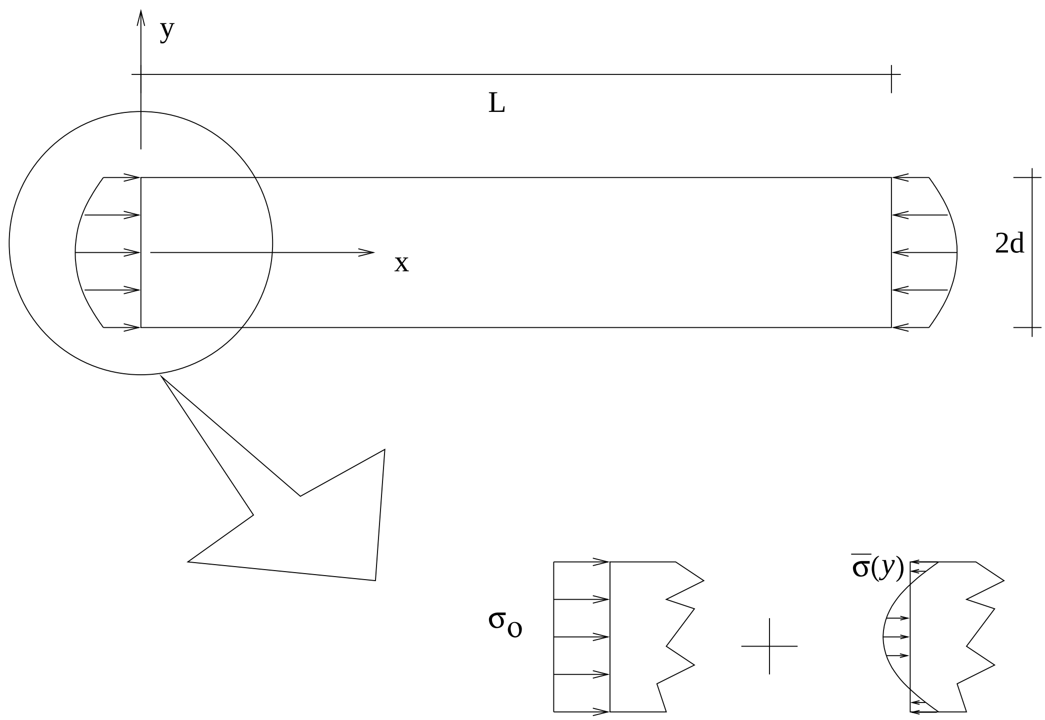

Figure 1: End-loaded bar with a uniform end traction plus a self equilibrated loading pattern.

We know that our Ansatz must also satisfy the bi-harmonic equation. If we plug our assumption into $\nabla^4\phi = 0$, we find that:

$$(\gamma/d)^4 f(y) + 2(\gamma/d)^2 f''(y) + f''''(y) = 0\,. \tag{11}$$

This is a 4th order ordinary differential equation with constant coefficients. Thus its solution is known to be of the form $f(y) = A\exp[sy]$. Inserting into (11) yields the following polynomial for s:

$$(\gamma/d)^4 + 2(\gamma/d)^2 s^2 + s^4 = 0\,. \tag{12}$$

This polynomial has only 2 unique roots $\pm i(\gamma/d)$; thus, we need to use variation of parameters to generate two more linearly independent solution (of the form $Ay\exp[sy]$). Combining these two additional solutions together with our first two solutions, we can write the functional form for our unknown function in our Ansatz as:

$$\begin{aligned} f(y) &= A\cosh(i\gamma y/d) + B\sinh(i\gamma y/d) \\ &\quad + Cy\cosh(i\gamma y/d) + Dy\sinh(i\gamma y/d)\,. \end{aligned} \tag{13}$$

In order to satisfy the lateral boundary conditions on σ_{yy} and σ_{yx} for all values of x, we must have that $f(\pm d) = 0$ and that $f'(\pm d) = 0$. These 4 conditions give us four equations for the 4 constants $A,\ B,\ C$ and D:

$$\begin{bmatrix} \cosh(i\gamma) & \sinh(i\gamma) & d\cosh(i\gamma) & d\sinh(i\gamma) \\ \cosh(i\gamma) & -\sinh(i\gamma) & -d\cosh(i\gamma) & d\sinh(i\gamma) \\ i\frac{\gamma}{d}\sinh(i\gamma) & i\frac{\gamma}{d}\cosh(i\gamma) & \cosh(i\gamma) + i\gamma\sinh(i\gamma) & \sinh(i\gamma) + i\gamma\cosh(i\gamma) \\ -i\frac{\gamma}{d}\sinh(i\gamma) & i\frac{\gamma}{d}\cosh(i\gamma) & \cosh(i\gamma) + i\gamma\sinh(i\gamma) & -\sinh(i\gamma) - i\gamma\cosh(i\gamma) \end{bmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\,. \tag{14}$$

If these equations are to have a non-trivial solution, then the determinant of the matrix in (14) must equal zero. The determinant of this matrix, after some lengthy but straightforward algebra, can be found to be:

$$\det[\cdot] = 2\gamma + \sin(2\gamma) = 0\,. \tag{15}$$

Equation (15) gives a relation that γ must satisfy in order for the lateral boundary conditions to hold true. Equation (15) has multiple complex-valued roots, the first two of which are:

$$\gamma_{1,2} = 2.1061 \pm i1.1254\,. \tag{16}$$

Remarks:

1. The first thing that we see is that the stresses associated with these roots will decay very rapidly as one moves away from the end of the bar. In particular,

$$\sigma^c_{(\cdot\cdot)} \sim \exp[-2.1061x/d]\,\{\cos(1.1254x/d) \pm i\sin(1.1254x/d)\}\,g(y)\,, \tag{17}$$

   where $g(y)$ is either $f(y)$, $f'(y)$ or $f''(y)$ depending upon the stress component. In numerical terms when $x/d = 1$ the influence of the correction stress is just 12% and for $x/d = 2$ the influence is only 1.5%.

2. Equation (16) has further roots all of which have $\mathrm{Re}[\gamma] > 2.1061$. Thus these additional roots give rise to local effects that decay even faster than those from (16). The next two roots, for example, are $\gamma_{3,4} = 5.3536 \pm i1.5516$.

3. The complete solution to the problem, if desired, can be found by using linear combinations of the solutions associated with all the roots. This provides the needed degrees of freedom to satisfy the remaining boundary conditions.