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Definitions of Some Useful Spaces

1 Real Vector Space

A real vector space is a set V and an operation $+ : V \times V \rightarrow V$ such that

1. $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a} \qquad \forall \boldsymbol{a}, \boldsymbol{b} \in V$
2. $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) \qquad \forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V$
3. $\exists \mathbf{0} \in V$ such that $\boldsymbol{a} + \mathbf{0} = \boldsymbol{a} \quad \forall \boldsymbol{a} \in V$
4. $\exists - \boldsymbol{a} \in V$ such that $\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0} \quad \forall \boldsymbol{a} \in V$

Further for any $\alpha \in \mathbb{R}$ and any vector $\boldsymbol{a} \in V$ their product $\alpha \boldsymbol{a}$ is in V and the following properties hold:

1. $1\boldsymbol{a} = \boldsymbol{a} \qquad \forall \boldsymbol{a} \in V$
2. $\alpha(\beta \boldsymbol{a}) = (\alpha\beta)\boldsymbol{a} \qquad \forall \boldsymbol{a} \in V \text{ and } \forall \alpha, \beta \in \mathbb{R}$
3. $\alpha(\boldsymbol{a} + \boldsymbol{b}) = \alpha\boldsymbol{a} + \alpha\boldsymbol{b}$ and $(\alpha + \beta)\boldsymbol{a} = \alpha\boldsymbol{a} + \beta\boldsymbol{a} \qquad \forall \boldsymbol{a} \in V$ and $\forall \alpha, \beta \in \mathbb{R}$

The symbol $\forall$ is read as "forall", the symbol $\exists$ is read as "there exists", and the symbol $\in$ is read as "in". The notation $+ : V \times V \rightarrow V$ is read as saying the symbol $+$ is a mapping from V and V into V. $\mathbb{R}$ denotes the space of real numbers.

2 Norm vector space

A norm vector space is a vector space V with a norm $\|\cdot\| : V \rightarrow \mathbb{R}$ with the following properties:

1. $\|\boldsymbol{a}\| \geq 0 \quad \forall \boldsymbol{a} \in V$ and is zero if and only if (*iff*) $\boldsymbol{a} = \mathbf{0}$
2. $\|\alpha\boldsymbol{a}\| = |\alpha| \, \|\boldsymbol{a}\| \qquad \forall \alpha \in \mathbb{R} \text{ and } \forall \boldsymbol{a} \in V$
3. $\|\boldsymbol{a} + \boldsymbol{b}\| \leq \|\boldsymbol{a}\| + \|\boldsymbol{b}\| \qquad \forall \boldsymbol{a}, \boldsymbol{b} \in V$

3 Inner Product Space

An inner product space is a vector space V with an inner product $\cdot : V \times V \to \mathbb{R}$ such that

1. $(\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{c} = \boldsymbol{a}\cdot\boldsymbol{c}+\boldsymbol{b}\cdot\boldsymbol{c} \qquad \forall \boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in V$
2. $(\alpha\boldsymbol{a})\cdot\boldsymbol{b} = \alpha(\boldsymbol{a}\cdot\boldsymbol{b}) \qquad \forall\alpha\in\mathbb{R}\,\text{and}\,\forall\boldsymbol{a},\boldsymbol{b}\in V$
3. $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a} \qquad \forall\boldsymbol{a},\boldsymbol{b}\in V$
4. $\boldsymbol{a}\cdot\boldsymbol{a}\geq 0 \quad \forall\boldsymbol{a}\in V$ and is zero *iff* $\boldsymbol{a}=\boldsymbol{0}$

Note that one can make an inner product space be a norm space by defining the "natural norm" $\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$.

4 Euclidean Vector Space

An Euclidean vector space is a vector space V with a (real valued) inner product, its natural norm, and a vector product $\wedge : V\times V\to V$ such that the following properties hold:

1. $\boldsymbol{a}\wedge\boldsymbol{b} = -\boldsymbol{b}\wedge\boldsymbol{a} \qquad \forall\boldsymbol{a},\boldsymbol{b}\in V$
2. $(\alpha\boldsymbol{a}+\beta\boldsymbol{b})\wedge\boldsymbol{c} = \alpha\boldsymbol{a}\wedge\boldsymbol{c}+\beta\boldsymbol{b}\wedge\boldsymbol{c} \qquad \forall\alpha,\beta\in\mathbb{R}\,\text{and}\,\forall\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in V$
3. $\boldsymbol{a}\cdot(\boldsymbol{a}\wedge\boldsymbol{b}) = 0 \qquad \forall\boldsymbol{a},\boldsymbol{b}\in V$
4. $(\boldsymbol{a}\wedge\boldsymbol{b})\cdot(\boldsymbol{a}\wedge\boldsymbol{b}) = (\boldsymbol{a}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{b})-(\boldsymbol{a}\cdot\boldsymbol{b})^2 \qquad \forall\boldsymbol{a},\boldsymbol{b}\in V$

The vector product we are most familiar with is the standard cross-product which is often denoted by the alternate symbol $\times$.