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Fitting Prony's Series

A Prony series is a series of the form $f(t) = \sum_{j=1}^{n} A_j \exp[\lambda_j t]$. In viscoelasticity this is the canonical form for the relaxation function and with a relaxation test the objective is to determine the constants A_j and λ_j from measured values of $f(t)$ at fixed moments in time. The problem is particularly difficult in that the λ_j appear non-linearly in the expression. Further the appropriate number n is also unknown.

One common method of fitting a Prony series is to fix a set of λ_j with a predetermined number of terms n. Then knowing $f(t)$ at a set of measured times, one can solve a set of linear equations to determine the A_j. This is a very easy, fast, and stable method but requires some intuition and understanding to get a reasonable result.

A second method (though sometimes a bit temperamental) is due to Prony himself. Suppose we know

$$f(0),\ f(\Delta t),\ f(2\Delta t), \ldots$$

In Prony's method we will first find $\alpha_j = \exp[\lambda_j \Delta t]$ (from which $\lambda_j = \ln[\alpha_j]/\Delta t$) and then we will find A_j. The procedure is to first note that

$$f(k\Delta t) = \sum A_j \alpha_j^k \qquad (k = 0, 1, 2, \ldots) \tag{1}$$

and that the α_j can be considered the roots of the polynomial

$$p(\alpha) = \prod_{j=1}^{n} (\alpha - \alpha_j) = \alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_0 \,.$$

With these relations one can observe that

$$\begin{array}{llr} f(0)c_0 + f(\Delta t)c_1 & + \cdots + f((n-1)\Delta t)c_{n-1} & + f(n\Delta t) = 0 \\ f(\Delta t)c_0 + f(2\Delta t)c_1 & + \cdots + f(n\Delta t)c_{n-1} & + f((n+1)\Delta t) = 0 \\ & \vdots & \\ f(k\Delta t)c_0 + f((k+1)\Delta t)c_1 & + \cdots + f((k+n-1)\Delta t)c_{n-1} & + f((k+n)\Delta t) = 0 \,. \end{array}$$

The first of these follow easily by expansion and noting that each α_j is a root of $p(\alpha) = 0$; the second follows by introducing the $\bar{A}_j = A_j\alpha_j$, expanding, and using the root property again; the third follows by a similar procedure. Rewriting in matrix vector form, we have:

$$\begin{bmatrix} f(0) & f(\Delta t) & \cdots & f((n-1)\Delta t) \\ f(\Delta t) & f(2\Delta t) & \cdots & f(n\Delta t) \\ & & \vdots & \\ f((n-1)\Delta t) & f(n\Delta t) & \cdots & f((2n-2)\Delta t) \end{bmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = - \begin{pmatrix} f(n\Delta t) \\ f((n+1)\Delta t) \\ \vdots \\ f((2n-1)\Delta t) \end{pmatrix}$$

With these equations we have a set of linear equations that we can solve for the c_j. One can use only the number of equations needed to uniquely determine the c_j (as shown) or one can use more equations and over determine the system – thus leading to a least square solution. Once the c_j are known, the roots α_j of $p(\alpha) = 0$ can be determined. Once these are known one can solve the system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1^1 & \alpha_2^1 & \cdots & \alpha_n^1 \\ & & \vdots & \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} f(0) \\ f(\Delta t) \\ \vdots \\ f((n-1)\Delta t) \end{pmatrix}$$

for the A_j. These relations are the first n equations from (1). Note that in this procedure one needs to select n ahead of time. By this method of two linear solves and a polynomial root finding, one can fit a Prony series. A nice discussion the method can be found in Hildebrand (1974, §9.4) as well as in Prony's original paper (Prony, 1795). It is noted however that the method can be a bit sensitive to experimental noise. So fixing the exponents, as first mentioned, is often much more robust. A third alternative, known as harmonic inversion, emanates from quantum chemistry and provides some balance between the methods highlighted here – see e.g. Wall and Neuhauser (1995) and Mandelshtam and Taylor (1997) as well as Johnson (2004).

References

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