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Infinite plate with a hole under far field tension

In these notes we will consider the problem of a very large plate with a central hole that is loaded in the far-field by a tensile stress field. To solve the problem we will consider doing it by superposition. First we will assume there is no hole in the plate and write down the solution. Then we will correct the solution by considering a second problem of a plate with a hole with no far-field stresses but a special traction distribution on the edge of the hole that will exactly cancel the solution from the plate without the hole. The sum of the two solutions will give a total solution that gives zero traction on the edge of the hole and far field tension.

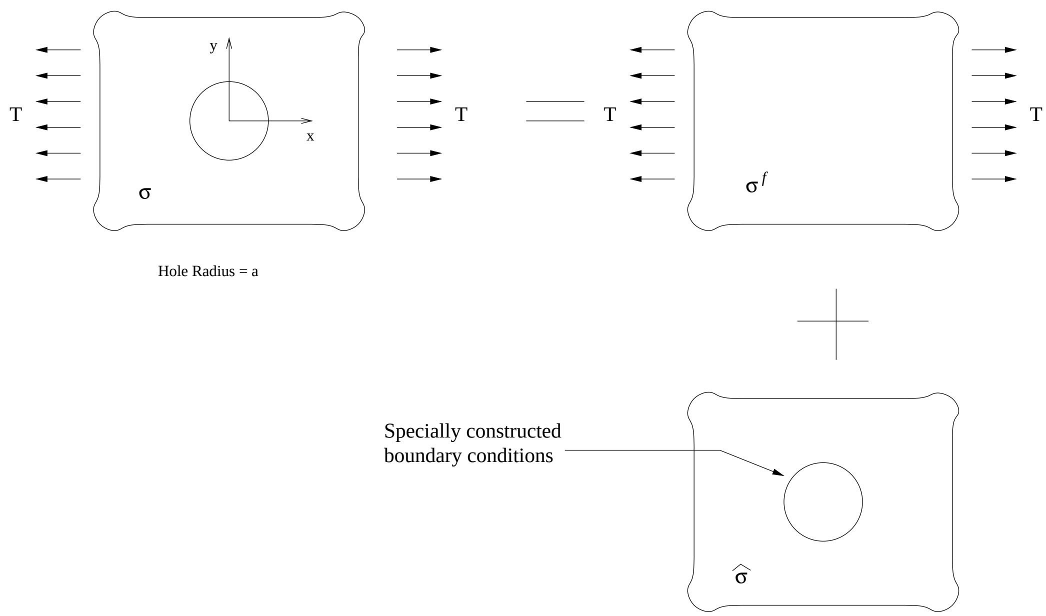

Figure 1: Infinite plate with a hole under far field tension decomposed into two problems.

The solution for the problem without the hole is given as

$$\boldsymbol{\sigma}^f = T\boldsymbol{e}_x \otimes \boldsymbol{e}_x \tag{1}$$

In polar coordinates this translates to

$$
\begin{aligned}
\sigma^f_{rr} &= \boldsymbol{e}_r \cdot \boldsymbol{\sigma} \boldsymbol{e}_r = T\cos^2(\theta) = \frac{T}{2}[1+\cos(2\theta)] && (2)\\
\sigma^f_{\theta\theta} &= \boldsymbol{e}_\theta \cdot \boldsymbol{\sigma} \boldsymbol{e}_\theta = T\sin^2(\theta) = \frac{T}{2}[1-\cos(2\theta)] && (3)\\
\sigma^f_{r\theta} &= \boldsymbol{e}_r \cdot \boldsymbol{\sigma} \boldsymbol{e}_\theta = -T\cos(\theta)\sin(\theta) = -\frac{T}{2}\sin(2\theta)\,. && (4)
\end{aligned}
$$

The only problem with using this stress field as the solution to the real problem is that it does not satisfy the traction free boundary condition at $r = a$. To correct this let us first compute the traction at $r = a$. The normal is $-\boldsymbol{e}_r$ so:

$$
\begin{aligned}
\boldsymbol{t}|_{r=a} = \boldsymbol{\sigma}^f|_{r=a}(-\boldsymbol{e}_r) &= -T\cos^2(\theta)\boldsymbol{e}_r + T\cos(\theta)\sin(\theta)\boldsymbol{e}_\theta \\
&= -\frac{T}{2}[1+\cos(2\theta)]\boldsymbol{e}_r + \frac{T}{2}\sin(2\theta)\boldsymbol{e}_\theta
\end{aligned} \tag{5}
$$

We now require that at $r = a$

$$
\hat{\boldsymbol{\sigma}}|_{r=a}(-\boldsymbol{e}_r) = \frac{T}{2}[1+\cos(2\theta)]\boldsymbol{e}_r - \frac{T}{2}\sin(2\theta)\boldsymbol{e}_\theta\,. \tag{6}
$$

In this way the sum of $\boldsymbol{\sigma}^f$ and $\hat{\boldsymbol{\sigma}}$ at $r = a$ will satisfy the zero traction boundary condition of the original problem. $\hat{\boldsymbol{\sigma}}$ also needs to satisfy a far-field boundary condition that it goes to zero as $r \to \infty$ since $\boldsymbol{\sigma}^f$ already satisfies the far-field boundary condition of the original problem. More specifically, we want that the force generated by $\hat{\boldsymbol{\sigma}}$ on an arc in the far-field go to zero as the radius of the arc becomes large. This implies that

$$
\lim_{r\to\infty} \int_{\text{arc}} \hat{\boldsymbol{\sigma}} r \, \mathrm{d}\theta = 0 \tag{7}
$$

This implies that $\hat{\boldsymbol{\sigma}}$ must go to zero at least as fast as $\frac{1}{r^k}$, where $k > 1$.

To solve for $\hat{\boldsymbol{\sigma}}$ we can use the Michell's general solution. Our traction boundary condition indicates that we will need terms that do not depend on θ and terms that depend upon 2θ. Thus we should for a start consider the following terms:

$$
\begin{aligned}
\Phi &= a_o \ln(r) + b_o r^2 + c_o r^2 \ln(r) + d_o r^2 \theta + a'_o \theta && (8)\\
&+ \left(a_2 r^2 + b_2 r^4 + a'_2/r^2 + b'_2\right)\cos(2\theta) && (9)\\
&+ \left(c_2 r^2 + d_2 r^4 + c'_2/r^2 + d'_2\right)\sin(2\theta) && (10)
\end{aligned}
$$

In (8) the term b_o does not give decaying stresses so it must be zero; c_o and d_o lead to non-single valued solutions for our geometry and thus they must be zero. In (9) and (10) the terms a_2, b_2, c_2, and d_2 do not give decaying stresses so they must be zero. Points about decaying stresses follow directly from

$$\begin{aligned} \hat{\sigma}_{rr} &= \frac{1}{r}\Phi_{,r} + \frac{1}{r^2}\Phi_{,\theta\theta} && (11) \\ \hat{\sigma}_{\theta\theta} &= \Phi_{,rr} && (12) \\ \hat{\sigma}_{r\theta} &= \frac{1}{r^2}\Phi_{,\theta} - \frac{1}{r}\Phi_{,r\theta} && (13) \end{aligned}$$

Let us start with (8) and (9) and then if we can not satisfy the boundary conditions with these terms then we can try adding in the terms from (10). This gives

$$\begin{aligned} \hat{\sigma}_{rr} &= \frac{a_o}{r^2} - \frac{6a'_2}{r^4}\cos(2\theta) - \frac{4b'_2}{r^2}\cos(2\theta) && (14) \\ \hat{\sigma}_{\theta\theta} &= -\frac{a_o}{r^2} + \frac{6a'_2}{r^4}\cos(2\theta) && (15) \\ \hat{\sigma}_{r\theta} &= \frac{a'_o}{r^2} - \frac{6a'_2}{r^4}\sin(2\theta) - \frac{2b'_2}{r^2}\sin(2\theta) && (16) \end{aligned}$$

Let us now apply the boundary conditions to try and determine the coefficients. This gives:

$$\begin{aligned} -\left[\frac{a_o}{a^2} - \left(\frac{6a'_2}{a^4} + \frac{4b'_2}{a^2}\right)\cos(2\theta)\right] &= \frac{T}{2}[1+\cos(2\theta)] && (17) \\ -\left[\frac{a'_o}{a^2} - \left(\frac{6a'_2}{a^4} + \frac{2b'_2}{a^2}\right)\sin(2\theta)\right] &= -\frac{T}{2}\sin(2\theta) && (18) \end{aligned}$$

Due to the orthogonality of trigonometric functions we can match coefficients of the constant terms and of each trigonometric function. This gives four linear equations in the four unknowns. Solving yields

$$\begin{aligned} a'_o &= 0 && (19) \\ a_o &= -\frac{T}{2}a^2 && (20) \\ a'_2 &= -\frac{T}{4}a^4 && (21) \\ b'_2 &= \frac{T}{2}a^2 && (22) \end{aligned}$$

Since we have been able to satisfy all the boundary conditions there is no need to look at the terms in (10).

The final solution is given by:

$$\sigma_{rr} = \frac{T}{2}\left\{1-\left(\frac{a}{r}\right)^2+\left[1-4\left(\frac{a}{r}\right)^2+3\left(\frac{a}{r}\right)^4\right]\cos(2\theta)\right\} \tag{23}$$

$$\sigma_{\theta\theta} = \frac{T}{2}\left\{1+\left(\frac{a}{r}\right)^2-\left[1+3\left(\frac{a}{r}\right)^4\right]\cos(2\theta)\right\} \tag{24}$$

$$\sigma_{r\theta} = -\frac{T}{2}\left[1+2\left(\frac{a}{r}\right)^2-3\left(\frac{a}{r}\right)^4\right]\sin(2\theta) \tag{25}$$

One very important result of this computation is the value of the stress concentration around the hole. The tangential stresses around the hole are given by

$$\sigma_{\theta\theta}|_{r=a} = T[1-2\cos(2\theta)] \tag{26}$$

This is maximal at $\theta = \pm\frac{\pi}{2}$ which gives $\sigma_{\theta\theta}^{\max} = 3T$.