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Representation Theorem for Linear Operators on Finite Dimensional Vector Spaces

Theorem 1 (Linear Operator Representation Theorem) *Given a linear operator* $\boldsymbol{l} : V \rightarrow V$*, where* V *is a finite dimensional vector space with basis* $\{\boldsymbol{e}_j\}_{j=1}^n$*, there exists a unique 2nd order tensor* $\boldsymbol{L}$ *such that*

$$\boldsymbol{l}(\boldsymbol{a}) = \boldsymbol{L}\boldsymbol{a} \qquad \forall \boldsymbol{a} \in V\,.$$

Proof: The proof of this result involves two issues: (1) the proof of existance and (2) the proof of uniqueness. Starting with issue (1), condsider an arbitrary $\boldsymbol{a} \in V$. Then one can write

$$\boldsymbol{a} = a_i \boldsymbol{e}_i\,.$$

Thus

$$\begin{aligned} \boldsymbol{l}(a_i\boldsymbol{e}_i) &= a_i \boldsymbol{l}(\boldsymbol{e}_i) \\ &= a_i l_j(\boldsymbol{e}_i)\boldsymbol{e}_j \\ &= [l_j(\boldsymbol{e}_k)\boldsymbol{e}_j \otimes \boldsymbol{e}_k] a_i \boldsymbol{e}_i \end{aligned}$$

Thus we can identify $\boldsymbol{L} = l_j(\boldsymbol{e}_k)\boldsymbol{e}_j \otimes \boldsymbol{e}_k$.
To prove uniqueness, assume that there are two such $\boldsymbol{L}$'s called $\boldsymbol{L}^1$ and $\boldsymbol{L}^2$. This implies for any $\boldsymbol{a} \in V$ that

$$\boldsymbol{l}(\boldsymbol{a}) = \boldsymbol{L}^1\boldsymbol{a} \qquad \text{and} \qquad \boldsymbol{l}(\boldsymbol{a}) = \boldsymbol{L}^2\boldsymbol{a}$$

Subtracting gives

$$\boldsymbol{0} = (\boldsymbol{L}^1 - \boldsymbol{L}^2)\boldsymbol{a} \qquad \forall \boldsymbol{a} \in V\,,$$

which implies that $\boldsymbol{L}^1 = \boldsymbol{L}^2$; q.e.d.