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Linearization of the Determinant

Consider the linearization of $\det(\boldsymbol{F})$ in the direction $\boldsymbol{H}$ at a "point" $\bar{\boldsymbol{F}}$. This is the function

$$\det(\bar{\boldsymbol{F}} + \boldsymbol{H}) \approx \det(\bar{\boldsymbol{F}}) + \left.\frac{d}{d\theta}\right|_{\theta=0} \det(\bar{\boldsymbol{F}} + \theta\boldsymbol{H})\,, \tag{1}$$

where the right-hand-side is the desired linearization. The entire issue is the computation of the last term on the right-hand-side of Eq. (1). To this end we make use of the following result from linear algebra

$$\det(\boldsymbol{A} - \lambda\boldsymbol{1}) = -\lambda^3 + \mathrm{tr}(\boldsymbol{A})\lambda^2 - \frac{1}{2}[\mathrm{tr}(\boldsymbol{A})^2 - \mathrm{tr}(\boldsymbol{A}^2)]\lambda + \det(\boldsymbol{A})\,. \tag{2}$$

Rewrite the last term on the right-hand-side of Eq. (1) as

$$\begin{aligned}
\left.\frac{d}{d\theta}\right|_{\theta=0} \det(\bar{\boldsymbol{F}} + \theta\boldsymbol{H}) &= \left.\frac{d}{d\theta}\right|_{\theta=0} \det((\boldsymbol{1} + \theta\boldsymbol{H}\bar{\boldsymbol{F}}^{-1})\bar{\boldsymbol{F}}) && (3)\\
&= \left.\frac{d}{d\theta}\right|_{\theta=0} \Big(\det(\bar{\boldsymbol{F}})\det(\boldsymbol{1} + \theta\boldsymbol{H}\bar{\boldsymbol{F}}^{-1})\Big) && (4)\\
&= \left.\frac{d}{d\theta}\right|_{\theta=0} \Big(\det(\bar{\boldsymbol{F}})\det(\theta\boldsymbol{H}\bar{\boldsymbol{F}}^{-1} - (-1)\boldsymbol{1})\Big) && (5)
\end{aligned}$$

Now apply Eq. (2) to the last determinant expression in Eq. (5) noting that $\boldsymbol{A} = \theta\boldsymbol{H}\bar{\boldsymbol{F}}^{-1}$ and $\lambda = -1$. Thus,

$$\begin{aligned}
\left.\frac{d}{d\theta}\right|_{\theta=0} \det(\bar{\boldsymbol{F}} + \theta\boldsymbol{H}) &= \left.\frac{d}{d\theta}\right|_{\theta=0} \Big(\det(\bar{\boldsymbol{F}})(1 + \mathrm{tr}(\theta\boldsymbol{H}\bar{\boldsymbol{F}}^{-1}) + o(\theta))\Big) && (6)\\
&= \det(\bar{\boldsymbol{F}})\mathrm{tr}(\boldsymbol{H}\bar{\boldsymbol{F}}^{-1}) && (7)
\end{aligned}$$

With this result in hand we can now linearize the Jacobian for the case of small deformations. Begin by considering a given state of deformation $\bar{\boldsymbol{x}}$ and a direction $\boldsymbol{u}$. (Later we will set $\bar{\boldsymbol{x}} = \boldsymbol{X}$; i.e. no deformation). Let

$$\boldsymbol{x}^\theta = \bar{\boldsymbol{x}} + \theta\boldsymbol{u}\,. \tag{8}$$

Thus

$$\boldsymbol{F}^\theta = \frac{\partial \boldsymbol{x}^\theta}{\partial \boldsymbol{X}} = \frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{X}} + \frac{\partial(\theta \boldsymbol{u})}{\partial \boldsymbol{X}} = \bar{\boldsymbol{F}} + \theta \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} \,. \tag{9}$$

Continuing, let

$$J^\theta = \det(\boldsymbol{F}^\theta) \,. \tag{10}$$

If we consider J as a function of the given positions $\bar{\boldsymbol{x}}$, then the linearization of J about the state $\bar{\boldsymbol{x}}$ in the direction $\boldsymbol{u}$ is given by the right hand side of

$$J(\bar{\boldsymbol{x}} + \boldsymbol{u}) \approx J(\bar{\boldsymbol{x}}) + \left.\frac{d}{d\theta}\right|_{\theta=0} J^\theta = \bar{J} + \left.\frac{d}{d\theta}\right|_{\theta=0} J^\theta \tag{11}$$

$$= \bar{J} + \left.\frac{d}{d\theta}\right|_{\theta=0} \det\left(\bar{\boldsymbol{F}} + \theta \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right) \,. \tag{12}$$

Identifying $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}$ with $\boldsymbol{H}$ from Eq. (7) yields

$$J(\bar{\boldsymbol{x}} + \boldsymbol{u}) \approx \bar{J} + \bar{J}\mathrm{tr}\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} \bar{\boldsymbol{F}}^{-1}\right) = \bar{J} + \bar{J}\mathrm{tr}\left(\frac{\partial \boldsymbol{u}}{\partial \bar{\boldsymbol{x}}}\right) \,. \tag{13}$$

If we now let the given state $\bar{\boldsymbol{x}}$ be equal to $\boldsymbol{X}$; i.e. we consider a linearization about the undeformed or reference state then $\bar{\boldsymbol{x}} = \boldsymbol{X}$, $\bar{J} = \det(\mathbf{1}) = 1$, and

$$J \approx 1 + \mathbf{1} : \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = 1 + \mathrm{div}[\boldsymbol{u}] \,. \tag{14}$$

Thus the linearized (for small deformations) change in volume is given by

$$V = (1 + \mathbf{1} : \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}) V_0 \,. \tag{15}$$

In a relative sense,

$$\frac{V - V_0}{V_0} = \mathbf{1} : \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}} = u_{i,i} = \varepsilon_{ii} = \mathrm{tr}(\boldsymbol{\varepsilon}) \,, \tag{16}$$

where the displacement gradients have been assumed small so that the distinction between derivatives with respect to $\boldsymbol{x}$ and $\boldsymbol{X}$ are of higher order than the case being considered.