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Higher Order Determinants

The determinant of a second order tensor can be expressed in several ways. Suppose $\boldsymbol{T}$ is a second order tensor in *n-dimensions*, then for a given coordinate system

$$\det[\boldsymbol{T}] = T_{1i_1}T_{2i_2}\cdots T_{ni_n}e_{i_1i_2\cdots i_n}\,, \tag{1}$$

where $i_1, i_2, \cdots, i_n$ are indices that range over the set $\{1, 2, 3, \cdots, n\}$. The value of the permutation symbol[1] is $+1$ if $(i_1, i_2, \cdots, i_n)$ form an even permutation and -1 if they form and odd permutation; otherwise, it is zero. Other expressions for the determinant are

$$\det[\boldsymbol{T}]e_{j_1j_2\cdots j_n} = T_{j_1i_1}T_{j_2i_2}\cdots T_{j_ni_n}e_{i_1i_2\cdots i_n} \tag{2}$$

and

$$\det[\boldsymbol{T}] = \frac{1}{n!}T_{j_1i_1}T_{j_2i_2}\cdots T_{j_ni_n}e_{i_1i_2\cdots i_n}e_{j_1j_2\cdots j_n}\,, \tag{3}$$

where $j_1, j_2, \cdots j_n$ are indices that range over the set $\{1, 2, 3, \cdots, n\}$. In order to use the expressions above one needs a definition of an even and odd permutation. To determine whether or not a sequence $(i_1, i_2, \cdots, i_n)$ is an even permutation or an odd permutation, one begins with the sequence $(1, 2, 3, \cdots, n)$ and counts the number of interchanges of numbers in the sequence required to generate $(i_1, i_2, \cdots, i_n)$. If the number of interchanges is an even number then the permutation is even if the number is odd then the permutation is odd. As an example the permutation $(i_1, i_2, \cdots, i_n) = (1, 4, 5, 3, 2)$ can be created by the following series of interchanges

$$(1, 2, 3, 4, 5) \rightarrow (1, 5, 3, 4, 2) \rightarrow (1, 5, 4, 3, 2) \rightarrow (1, 4, 5, 3, 2)\,. \tag{4}$$

Thus there are 3 interchanges and the permutation is considered odd. This implies that $e_{14532} = -1$. All these formulae readily reduce to the expressions given in class for the case of $n = 3$.

[1] Other names for the permutation symbol are alternating symbol and Levi-Civita symbol. Some texts consider the permutation symbol to be the components of a tensor and others do not; the issue is the precise definition of a tensor used in different texts. For our purposes the idea of the permutation symbol will be sufficient and we will not need to consider its tensorial properties or lack thereof .

Advanced references:

1. D. Lovelock and H. Rund, *Tensors, Differential Forms, and Variational Principles*, Dover, 1989.

2. W.L. Burke, *Applied Differential Geometry*, Cambridge Press, 1985.