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**Miscellaneous Linear Algebra:**
**Eigenvalues and Eigenvectors**

# 1 Definition of Eigenvalues and Eigenvectors

Given a tensor $\boldsymbol{A}$ the scalar $\lambda$ and the vector $\boldsymbol{p}$ which satisfy

$$\boldsymbol{A}\boldsymbol{p} = \lambda \boldsymbol{p}\,, \tag{1}$$

are known as the eigenvalue and eigenvector of $\boldsymbol{A}$. Note that these "eigenpairs" are not unique. As means of finding the eigenpairs note that (1) can be written as

$$(\boldsymbol{A} - \lambda \mathbf{1})\boldsymbol{p} = \mathbf{0}\,. \tag{2}$$

The only way for (2) to have a non-trivial solution is for the tensor in the parenthesis to have zero determinant. Thus, the governing equation for the eigenvalues is given by

$$\det[\boldsymbol{A} - \lambda \mathbf{1}] = 0\,. \tag{3}$$

This can be expanded to give a (characteristic) polynomial in $\lambda$ of $n^{\text{th}}$ order where $n$ is the order of $\boldsymbol{A}$. For the case of second order tensors this expansion gives

$$-\lambda^3 + I_A \lambda^2 - II_A \lambda + III_A = 0\,, \tag{4}$$

where $I_A = \text{tr}[\boldsymbol{A}] = \lambda_1 + \lambda_2 + \lambda_3$, $II_A = \frac{1}{2}\{(\text{tr}[\boldsymbol{A}])^2 - \text{tr}[\boldsymbol{A}^2]\} = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$, and $III_A = \det[\boldsymbol{A}] = \lambda_1\lambda_2\lambda_3$ are the 3 invariants of $\boldsymbol{A}$. Solving this polynomial for the $\lambda$'s gives the eigenvalues of $\boldsymbol{A}$. Once the eigenvalues are known, Eq. (2) can be used to determine the corresponding eigenvectors. Note that the vectors are not unique. By convention we will always normalize them to have unit length.

# 2 Eigenvalues are real for $\boldsymbol{\varepsilon} \in \mathbb{S}^3$

Eigenvalues are real for a symmetric real valued tensors. Proof: Assume that $\lambda$ and $\boldsymbol{n}$ are an Eigenpair; i.e.

$$\boldsymbol{\varepsilon}\boldsymbol{n} = \lambda \boldsymbol{n}\,. \tag{5}$$

Since $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T$ this implies that $\boldsymbol{\varepsilon}^T \boldsymbol{n} = \lambda \boldsymbol{n}$ or that

$$\bar{\boldsymbol{\varepsilon}}^T \boldsymbol{n} = \lambda \boldsymbol{n} \tag{6}$$

where the bar indicates complex conjugation. Dot (6) by $\bar{\boldsymbol{n}}$ so that

$$\bar{\boldsymbol{n}} \cdot \bar{\boldsymbol{\varepsilon}}^T \boldsymbol{n} = \lambda \bar{\boldsymbol{n}} \cdot \boldsymbol{n}\,. \tag{7}$$

Conjugate (5) and dot by $\boldsymbol{n}$ so that

$$\boldsymbol{n} \cdot \bar{\boldsymbol{\varepsilon}} \bar{\boldsymbol{n}} = \bar{\lambda} \boldsymbol{n} \cdot \bar{\boldsymbol{n}}\,. \tag{8}$$

Apply the definition of transpose to (8) to give

$$\bar{\boldsymbol{\varepsilon}}^T \boldsymbol{n} \cdot \bar{\boldsymbol{n}} = \bar{\lambda} \boldsymbol{n} \cdot \bar{\boldsymbol{n}}\,. \tag{9}$$

Compare (9) to (7) to reveal that

$$\lambda \bar{\boldsymbol{n}} \cdot \boldsymbol{n} = \bar{\lambda} \boldsymbol{n} \cdot \bar{\boldsymbol{n}}\,. \tag{10}$$

Thus we have that $\bar{\lambda} = \lambda$ which implies that $\lambda$ is a real number.

# 3 Eigenvector are orthogonal for $\boldsymbol{\varepsilon} \in \mathbb{S}^3$

Proof of orthogonality of eigenvectors: Let $(\lambda_1, \boldsymbol{x}^1)$ and $(\lambda_2, \boldsymbol{x}^2)$ be two eigenpairs for $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T$ (real) where $\lambda_1 \neq \lambda_2$. Then $\boldsymbol{\varepsilon}\boldsymbol{x}^1 = \lambda_1 \boldsymbol{x}^1$ and $\boldsymbol{\varepsilon}\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}^2$. Dot these two expression by $\boldsymbol{x}^2$ and $\boldsymbol{x}^1$ respectively. Thus

$$\boldsymbol{x}^2 \cdot \boldsymbol{\varepsilon}\boldsymbol{x}^1 = \lambda_1 \boldsymbol{x}^2 \cdot \boldsymbol{x}^1 \tag{11}$$

and

$$\boldsymbol{x}^1 \cdot \boldsymbol{\varepsilon}\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}^1 \cdot \boldsymbol{x}^2\,. \tag{12}$$

Apply the definition of transpose and invoke the relation $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T$ on (11) to give

$$\boldsymbol{\varepsilon}^T \boldsymbol{x}^2 \cdot \boldsymbol{x}^1 = \boldsymbol{\varepsilon}\boldsymbol{x}^2 \cdot \boldsymbol{x}^1 = \lambda_1 \boldsymbol{x}^2 \cdot \boldsymbol{x}^1\,. \tag{13}$$

Combining (13) with (12) yields

$$(\lambda_1 - \lambda_2)\boldsymbol{x}^2 \cdot \boldsymbol{x}^1 = 0\,. \tag{14}$$

This implies that $\boldsymbol{x}^1 \cdot \boldsymbol{x}^2 = 0$ since the eigenvalues were assumed distinct.

# 4 Spectral Representations

For real symmetric 2nd order tensor, $\boldsymbol{A}$, one has the following useful representation theorems. If one has distinct eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ with associated eigenvectors $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3\}$ then we can write

$$\boldsymbol{A} = \sum_{i=1}^{3} \lambda_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i \,. \tag{15}$$

The proof of this form follow easily if one notes that the identity can be expressed in terms of the eigenvectors as $\mathbf{1} = \boldsymbol{n}_i \otimes \boldsymbol{n}_i$.
If we have $\lambda_1, \lambda_2 = \lambda_3$ with $\boldsymbol{n}_1$ the eigenvector associated with $\lambda_1$ , then we have

$$\boldsymbol{A} = \lambda_1 \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 + \lambda_2 (\mathbf{1} - \boldsymbol{n}_1 \otimes \boldsymbol{n}_1) \,. \tag{16}$$

Note that $\mathbf{1} - \boldsymbol{n}_1 \otimes \boldsymbol{n}_1$ represents a projection onto the subspace orthogonal to $\boldsymbol{n}_1$ and thus represents a plane of eigenvectors. If we have $\lambda_1 = \lambda_2 = \lambda_3$, then

$$\boldsymbol{A} = \lambda_1 \mathbf{1} \tag{17}$$

and all vectors are eigenvectors.

# 5 Caley-Hamilton Theorem

The Caley-Hamilton Theorem states that a tensor satisfies its own characteristic polynomial. For simplicity we will only deal with the real-symmetric case. To prove this we first start with an auxiliary result.
Assume that $f(\cdot)$ is a real polynomial and that $\hat{\lambda}$ is a solution of the characteristic polynomial for a tensor $\boldsymbol{A}$, then $f(\hat{\lambda})$ is an eigenvalue of $f(\boldsymbol{A})$. The proof is performed via the principle of mathematical induction (PMI). First note that there exists a $\boldsymbol{p}$ such that $\boldsymbol{A}^r \boldsymbol{p} = \hat{\lambda}^r \boldsymbol{p}$ for $r = 1$ (by assumption). Now assume

$$\boldsymbol{A}^n \boldsymbol{p} = \hat{\lambda}^n \boldsymbol{p} \tag{18}$$

for some fixed $n > 1$. Thus

$$\boldsymbol{A}^{n+1} \boldsymbol{p} = \boldsymbol{A}(\boldsymbol{A}^n \boldsymbol{p}) = \boldsymbol{A} \hat{\lambda}^n \boldsymbol{p} = \hat{\lambda}^n \boldsymbol{A} \boldsymbol{p} = \hat{\lambda}^n \hat{\lambda} \boldsymbol{p} = \hat{\lambda}^{n+1} \boldsymbol{p} \,. \tag{19}$$

Thus by PMI we have that $\boldsymbol{A}^r \boldsymbol{p} = \hat{\lambda}^r \boldsymbol{p}$ for all $r$. The remainder of the proof follow directly by expansion. $\square$

To show the main result, assume that the polynomial $f(\cdot)$ is the characteristic polynomial. Then $f(\hat{\lambda})$ is an eigenvalue of

$$f(\boldsymbol{A}) = -\boldsymbol{A}^3 + I_A\boldsymbol{A}^2 - II_A\boldsymbol{A} + III_A\,. \tag{20}$$

But $f(\hat{\lambda}) \equiv 0$ by our choice of $f(\cdot)$. This implies that all the eigenvalues of $f(\boldsymbol{A})$ are zero. For real symmetric tensors, the only such tensor is the zero tensor (by spectral representation) so we have the final result:

$$-\boldsymbol{A}^3 + I_A\boldsymbol{A}^2 - II_A\boldsymbol{A} + III_A = \boldsymbol{0}\,. \tag{21}$$

Note that the result is more general than the symmetric-real case, (it works for all tensors), but the proof is a bit more involved. See Gilbert Strang *Linear Algebra and Its Applications* for a proof of the more general result utilizing Schur's Lemma.

# 6 Max-min properties of eigenvalues

If we consider the strain tensor $\boldsymbol{\varepsilon}$, then the eigenvalues are the max, min, and "saddle point" normal strains over all directions. Proof: Let $\varepsilon_I$ be the Eigenvalues of $\boldsymbol{\varepsilon}$ and $\boldsymbol{n}^I$ the corresponding Eigenvectors. Find $\boldsymbol{v}$ such that

$$\boldsymbol{v}\cdot\boldsymbol{\varepsilon}\boldsymbol{v} \tag{22}$$

is a critical value (min, max, or saddle point), where $\boldsymbol{v}$ is unit. First write $\boldsymbol{\varepsilon}$ in the basis defined by the eigenvectors

$$\boldsymbol{\varepsilon} = \sum_{I=1}^{3} \varepsilon_I \boldsymbol{n}^I \otimes \boldsymbol{n}^I\,. \tag{23}$$

Also assume that $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$. Then

$$\boldsymbol{v}\cdot\boldsymbol{\varepsilon}\boldsymbol{v} = \varepsilon_1(\boldsymbol{n}^1\cdot\boldsymbol{v})^2 + \varepsilon_2(\boldsymbol{n}^2\cdot\boldsymbol{v})^2 + \varepsilon_3(\boldsymbol{n}^3\cdot\boldsymbol{v})^2\,. \tag{24}$$

Let $(\boldsymbol{n}^1\cdot\boldsymbol{v}) = l$, $(\boldsymbol{n}^2\cdot\boldsymbol{v}) = m$, and $(\boldsymbol{n}^3\cdot\boldsymbol{v}) = n$ and note that $l^2+m^2+n^2 = 1$ since $\boldsymbol{v}$ is assumed to be unit. Therefore we need to find the critical values of $\varepsilon_1 l^2 + \varepsilon_2 m^2 + \varepsilon_3 n^2$ subject to the constraint that $l^2 + m^2 + n^2 = 1$. This is done via the method of Lagrange multipliers; form the Lagrangian

$$\mathcal{L} = \varepsilon_1 l^2 + \varepsilon_2 m^2 + \varepsilon_3 n^2 + \lambda(l^2 + m^2 + n^2 - 1)\,. \tag{25}$$

The critical equations of the Lagrangian are

$$\frac{\partial \mathcal{L}}{\partial l} = 0 \tag{26}$$

$$\frac{\partial \mathcal{L}}{\partial m} = 0 \tag{27}$$

$$\frac{\partial \mathcal{L}}{\partial n} = 0 \tag{28}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0\,. \tag{29}$$

Expanding the derivatives gives

$$\begin{aligned}(\varepsilon_1 + \lambda)l &= 0 \\ (\varepsilon_2 + \lambda)m &= 0 \\ (\varepsilon_3 + \lambda)n &= 0 \\ l^2 + m^2 + n^2 &= 1\,.\end{aligned} \tag{30}$$

To satisfy Eqs. (30) there are several choices:

1. $\lambda = -\varepsilon_1$, $m = 0$, $n = 0$, $l = 1$ which implies that $\boldsymbol{v} \cdot \boldsymbol{\varepsilon v} = \varepsilon_1$.
2. $\lambda = -\varepsilon_2$, $m = 1$, $n = 0$, $l = 0$ which implies that $\boldsymbol{v} \cdot \boldsymbol{\varepsilon v} = \varepsilon_2$.
3. $\lambda = -\varepsilon_3$, $m = 0$, $n = 1$, $l = 0$ which implies that $\boldsymbol{v} \cdot \boldsymbol{\varepsilon v} = \varepsilon_3$.

   *Remark:* Choice 1 leads to a maximum. Choice 3 leads to a minimum. And Choice 2 leads to a saddle point.□

Thus the primary conclusion is that $\varepsilon_1$ is the maximum normal strain and $\varepsilon_3$ is the minimum normal strain.