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Demonstration of the determinant relation

As discussed in lecture the volume ratio of a parallelepiped after and before deformation is given by:

$$\frac{V}{V_o} = \frac{(\boldsymbol{F} \cdot \boldsymbol{c}) \cdot ((\boldsymbol{F} \cdot \boldsymbol{a}) \times (\boldsymbol{F} \cdot \boldsymbol{b}))}{\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})}\,, \tag{1}$$

where the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are 'local' vectors to the point of interest in the body. To show that this ratio is indeed the determinant of $\boldsymbol{F}$, consider the three vectors to be of the form $\boldsymbol{a} = \epsilon_1 \boldsymbol{e}_1$, $\boldsymbol{b} = \epsilon_2 \boldsymbol{e}_2$, and $\boldsymbol{c} = \epsilon_3 \boldsymbol{e}_3$. In this case,

$$\begin{aligned} \frac{(\boldsymbol{F} \cdot \boldsymbol{c}) \cdot ((\boldsymbol{F} \cdot \boldsymbol{a}) \times (\boldsymbol{F} \cdot \boldsymbol{b}))}{\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})} &= \frac{\epsilon_1\epsilon_2\epsilon_3}{\epsilon_1\epsilon_2\epsilon_3} \frac{F_{kC} a_C e_{ijk} F_{iA} b_A F_{jB} c_B}{1} & (2) \\ &= e_{ijk} F_{i1} F_{j2} F_{j3}\,. & (3) \end{aligned}$$

The last expression is simply the indicial form for the determinant that we have already learned. It expresses the fundamental definition of a determinant, viz. that the determinant is the (signed) sum of all even and odd permutations of the products of elements of the rows of a matrix. At this point we have shown the result for 3 orthogonal vectors. To see that the result holds for 3 arbitrary vectors first note that:

$$\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})\,, \tag{4}$$

which implies that the scalar triple product of 3 vectors only depends upon the mutually orthogonal components of the 3 vectors and that is what we have already proved.