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A Quick Overview of Curvilinear Coordinates

1 Introduction

Curvilinear coordinate systems are general ways of locating points in Euclidean space using coordinate functions that are invertible functions of the usual x_i Cartesian coordinates. Their utility arises in problems with obvious geometric symmetries such as cylindrical or spherical symmetry. Thus our main interest in these notes is to detail the important relations for strain and stress in these two coordinate systems. Shown in Fig. 1 are the definitions of the coordinate functions. Note that while the definition of the cylindrical coordinate system is rather standard, the definition of the spherical coordinate system varies from book to book. Both systems to be studied are orthogonal. The precise definitions used here are:

Cylindrical

$$\begin{aligned} x_1 &= r\cos(\theta) \\ x_2 &= r\sin(\theta) \\ x_3 &= z \end{aligned} \tag{1}$$

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2} \\ \theta &= \tan^{-1}(x_2/x_1) \\ z &= x_3 \end{aligned} \tag{2}$$

Spherical

$$\begin{aligned} x_1 &= r\sin(\varphi)\cos(\theta) \\ x_2 &= r\sin(\varphi)\sin(\theta) \\ x_3 &= r\cos(\varphi) \end{aligned} \tag{3}$$

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2 + x_3^2} \\ \varphi &= \cos^{-1}\left(\frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}\right) \\ \theta &= \tan^{-1}(x_2/x_1) \end{aligned} \tag{4}$$

2 Basis Vectors

For convenience in some of the equations to be given later we will denote our curvilinear coordinates as z^k where $(z^1, z^2, z^3) = (r, \theta, z)$ in the cylindrical case and $(z^1, z^2, z^3) = (r, \varphi, \theta)$ in the spherical case. For the basis vectors we will introduce for two types of basis vectors. The natural basis vectors and

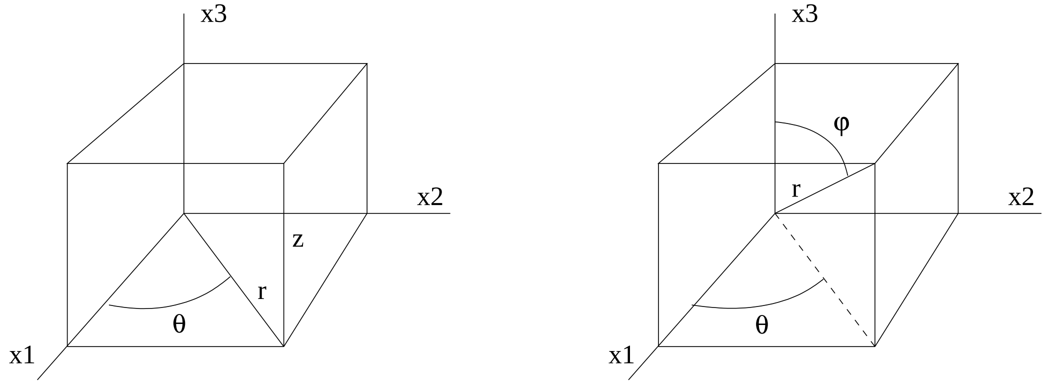

Figure 1: Definition of the cylindrical and spherical coordinate systems.

the physical basis vectors. Both bases are orthogonal but the physical basis has the additional property of orthonormality. The basic definitions are

$$\boldsymbol{g}_k = \frac{\partial x^i}{\partial z^k}\boldsymbol{e}_i \tag{5}$$

$$\boldsymbol{e}_k = \frac{\boldsymbol{g}_k}{\|\boldsymbol{g}_k\|}\,. \tag{6}$$

To differentiate between the physical basis vectors and the usual Cartesian ones we typically write $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \cdots$ etc. For the cylindrical coordinate system one has:

$$\boldsymbol{g}_1 \rightarrow \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} \qquad \boldsymbol{g}_2 \rightarrow \begin{pmatrix} -r\sin(\theta) \\ r\cos(\theta) \\ 0 \end{pmatrix} \qquad \boldsymbol{g}_3 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{7}$$

and

$$\boldsymbol{e}_r = \boldsymbol{g}_1 \qquad \boldsymbol{e}_\theta = \frac{1}{r}\boldsymbol{g}_2 \qquad \boldsymbol{e}_z = \boldsymbol{g}_3\,. \tag{8}$$

Note that the components have been expressed in the standard orthonormal Cartesian basis. For the spherical system one has that

$$\begin{aligned} \boldsymbol{g}_1 \rightarrow \begin{pmatrix} \sin(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\varphi) \end{pmatrix} \qquad & \boldsymbol{g}_2 \rightarrow \begin{pmatrix} r\cos(\varphi)\cos(\theta) \\ r\cos(\varphi)\sin(\theta) \\ -r\sin(\varphi) \end{pmatrix} \\ & \boldsymbol{g}_3 \rightarrow \begin{pmatrix} -r\sin(\varphi)\sin(\theta) \\ r\sin(\varphi)\cos(\theta) \\ 0 \end{pmatrix} \end{aligned} \tag{9}$$

and

$$\boldsymbol{e}_r = \boldsymbol{g}_1 \qquad \boldsymbol{e}_\varphi = \frac{1}{r}\boldsymbol{g}_2 \qquad \boldsymbol{e}_\theta = \frac{1}{r\sin(\varphi)}\boldsymbol{g}_3\,. \tag{10}$$

2.1 Physical and Natural Components

As with all bases we can express the components of vectors and tensors with respect to our new curvilinear bases. In this regard, it is very important with the curvilinear coordinates to know whether or not the components are with respect to the natural basis vectors or with respect to the physical basis vectors. To help maintain the distinction we use superscript numerals with the components in the natural basis and subscript letter (Latin and Greek) for components in the physical basis. Consider for example a vector $\boldsymbol{v} = v_\theta \boldsymbol{e}_\theta = v^2 \boldsymbol{g}_2$, then we have the relation

$$v_\theta = r v^2\,. \tag{11}$$

If for instance we have a vector $\boldsymbol{v} = v_\varphi \boldsymbol{e}_\varphi + v_\theta \boldsymbol{e}_\theta = v^2 \boldsymbol{g}_2 + v^3 \boldsymbol{g}_3$, then we have that

$$v_\varphi = r v^2 \tag{12}$$

$$v_\theta = r\sin(\varphi) v^3\,. \tag{13}$$

Similar relations can be derived for tensor components.

3 Dual Basis Vectors

When dealing with non-Cartesian coordinate systems one often introduces the so called dual (or contravariant) basis vectors; they are denoted by the symbol $\boldsymbol{g}^k$ – note the raised index. The defining property of these basis vectors is that they are orthogonal to the first basis introduced; i.e.

$$\boldsymbol{g}^i \cdot \boldsymbol{g}_j = \delta^i_j\,, \tag{14}$$

where δ^i_j is simply the Kronecker delta symbol. The i index is raised so that it matches the other side of the equation. The meaning is still the same (1 if $i = j$ and 0 otherwise). Another way of writing this is $\boldsymbol{g}^k = (\partial z^k/\partial x^i)\boldsymbol{e}^i$.

Note that for the usual Cartesian coordinates there is no difference between the dual basis and the regular basis. For the cylindrical system we have:

$$\boldsymbol{g}^1 \to \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} \qquad \boldsymbol{g}^2 \to \begin{pmatrix} -\sin(\theta)/r \\ \cos(\theta)/r \\ 0 \end{pmatrix} \qquad \boldsymbol{g}^3 \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{15}$$

Note again that the components have been expressed in the standard orthonormal Cartesian basis. For the spherical system one has that

$$\begin{aligned} \boldsymbol{g}^1 \to \begin{pmatrix} \sin(\varphi)\cos(\theta) \\ \sin(\varphi)\sin(\theta) \\ \cos(\varphi) \end{pmatrix} \qquad & \boldsymbol{g}^2 \to \begin{pmatrix} \cos(\varphi)\cos(\theta)/r \\ \cos(\varphi)\sin(\theta)/r \\ -\sin(\varphi)/r \end{pmatrix} \\ & \boldsymbol{g}^3 \to \begin{pmatrix} -\sin(\theta)/r\sin(\varphi) \\ \cos(\theta)/r\sin(\varphi) \\ 0 \end{pmatrix} . \end{aligned} \tag{16}$$

4 Gradient of a Scalar Function

Consider a scalar function f. Its gradient is given as ∇f. This can be converted through the use of the chain rule into curvilinear coordinates as:

$$\nabla f = \frac{\partial f}{\partial \boldsymbol{x}} = \frac{\partial f}{\partial x^i}\boldsymbol{e}^i = \frac{\partial f}{\partial z^k}\frac{\partial z^k}{\partial x^i}\boldsymbol{e}^i = \frac{\partial f}{\partial z^k}\boldsymbol{g}^k . \tag{17}$$

Typically, however, results are expressed using the physical basis vectors and not the natural basis vectors. For our two coordinates systems we have upon expansion:

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta + \frac{\partial f}{\partial z}\boldsymbol{e}_z \tag{18}$$

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\boldsymbol{e}_\varphi + \frac{1}{r\sin(\varphi)}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta \tag{19}$$

5 Gradient of a Vector

To compute the gradient of a vector expressed in curvilinear coordinates we need to be able to compute the gradient of the basis vectors as they are functions of position (unlike in the Cartesian case). Importantly we will need to know the derivatives

$$\frac{\partial \boldsymbol{g}_i}{\partial z^j} = \frac{\partial}{\partial z^j}\frac{\partial x^k}{\partial z^i}\boldsymbol{e}_k = \frac{\partial^2 x^k}{\partial z^j \partial z^i}\boldsymbol{e}_k \,. \tag{20}$$

The components of these vectors are usually expressed in the dual basis as

$$\Gamma^k_{ij} = \boldsymbol{g}^k \cdot \frac{\partial \boldsymbol{g}_i}{\partial z^j}\,, \tag{21}$$

where Γ^k_{ij} is called the Christoffel symbol. For the cylindrical coordinate system all of the Christoffel symbols are zero except

$$\Gamma^1_{22} = -r\,, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}\,. \tag{22}$$

For the spherical coordinate system we have all of the Christoffel symbols are zero except

$$\begin{aligned}
\Gamma^1_{22} &= -r\,, & \Gamma^1_{33} &= -r\sin^2(\varphi) \\
\Gamma^2_{12} = \Gamma^2_{21} &= \tfrac{1}{r}\,, & \Gamma^2_{33} &= -\sin(\varphi)\cos(\varphi) \\
\Gamma^3_{13} = \Gamma^3_{31} &= \tfrac{1}{r}\,, & \Gamma^3_{32} = \Gamma^3_{23} &= \cot(\varphi)
\end{aligned} \tag{23}$$

We can now consider taking the gradient of a vector. This gives

$$\nabla \boldsymbol{v} = \frac{\partial}{\partial \boldsymbol{x}}\left(v^i \boldsymbol{g}_i\right) = \frac{\partial v^i}{\partial \boldsymbol{x}}\boldsymbol{g}_i + v^i \frac{\partial \boldsymbol{g}_i}{\partial \boldsymbol{x}} \tag{24}$$

$$= \frac{\partial v^i}{\partial z^k}\boldsymbol{g}_i \otimes \boldsymbol{g}^k + v^i \frac{\partial \boldsymbol{g}_i}{\partial z^k} \otimes \boldsymbol{g}^k \tag{25}$$

$$= \left(\frac{\partial v^i}{\partial z^k} + +v^j \Gamma^i_{jk}\right)\boldsymbol{g}_i \otimes \boldsymbol{g}^k \,. \tag{26}$$

For the cylindrical coordinate system we can expand this result to determine the needed components of the gradient. When expressed in terms of the physical basis we find that $\nabla \boldsymbol{u}$ is given by

$$\begin{bmatrix} u_{r,r} & \frac{1}{r}(u_{r,\theta} - u_\theta) & u_{r,z} \\ u_{\theta,r} & \frac{1}{r}(u_{\theta,\theta} + u_r) & u_{\theta,z} \\ u_{z,r} & \frac{1}{r}u_{z,\theta} & u_{z,z} \end{bmatrix}. \tag{27}$$

The components of the symmetric part of this tensor give the strain, $\boldsymbol{\varepsilon}$, (in the physical basis as)

$$\begin{bmatrix} u_{r,r} & \frac{1}{2}\left(\frac{1}{r}u_{r,\theta} + r(u_\theta/r)_{,r}\right) & \frac{1}{2}(u_{r,z} + u_{z,r}) \\ & \frac{1}{r}(u_{\theta,\theta} + u_r) & \frac{1}{2}(u_{\theta,z} + \frac{1}{r}u_{z,\theta}) \\ \text{sym.} & & u_{z,z} \end{bmatrix}. \tag{28}$$

For the spherical coordinate system we can also expand this result to determine the needed components of the gradient. When expressed in terms of the physical basis we find that $\nabla \boldsymbol{u}$ is given by

$$\begin{bmatrix} u_{r,r} & \frac{1}{r}(u_{r,\varphi} - u_\varphi) & \frac{1}{r\sin(\varphi)}(u_{r,\theta} - u_\theta \sin(\varphi)) \\ u_{\varphi,r} & \frac{1}{r}(u_{\varphi,\varphi} + u_r) & -\frac{1}{r}u_\theta \cot(\varphi) + \frac{1}{r\sin(\varphi)}u_{\varphi,\theta} \\ u_{\theta,r} & \frac{1}{r}u_{\theta,\varphi} & \frac{1}{r\sin(\varphi)}u_{\theta,\theta} + u_r/r + u_\varphi \cot(\varphi)/r \end{bmatrix} \tag{29}$$

The components of the symmetric part of this tensor give the strain, $\boldsymbol{\varepsilon}$, (in the physical basis as)

$$\begin{bmatrix} u_{r,r} & \frac{1}{2}\left(\frac{1}{r}u_{r,\varphi} + r(u_\varphi/r)_{,r}\right) & \frac{1}{2}\left(\frac{1}{r\sin(\varphi)}u_{r,\theta} + r(u_\theta/r)_{,r}\right) \\ & \frac{1}{r}(u_{\varphi,\varphi} + u_r) & \frac{1}{2}\left(-\frac{1}{r}u_\theta \cot(\varphi) + \frac{1}{r}u_{\theta,\varphi} + \frac{1}{r\sin(\varphi)}u_{\varphi,\theta}\right) \\ \text{sym.} & & \frac{1}{r\sin(\varphi)}u_{\theta,\theta} + u_r/r + u_\varphi \cot(\varphi)/r \end{bmatrix} \tag{30}$$

6 Gradient and Divergence of a Tensor

The basic procedure for finding the gradient and divergence of a tensor follows exactly as we did above. For simplicity consider the stress tensor $\boldsymbol{\sigma}$. Its gradient is given by

$$\nabla \boldsymbol{\sigma} = \frac{\partial}{\partial \boldsymbol{x}}\left(\sigma^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j\right) = \frac{\partial \sigma^{ij}}{\partial \boldsymbol{x}}\boldsymbol{g}_i \otimes \boldsymbol{g}_j + \sigma^{ij}\frac{\partial \boldsymbol{g}_i}{\partial \boldsymbol{x}} \otimes \boldsymbol{g}_j + \sigma^{ij}\boldsymbol{g}_i \otimes \frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{x}} \tag{31}$$

$$= \frac{\partial \sigma^{ij}}{\partial z^k}\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}^k + \sigma^{ij}\frac{\partial \boldsymbol{g}_i}{\partial z^k} \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}^k + \sigma^{ij}\boldsymbol{g}_i \otimes \frac{\partial \boldsymbol{g}_j}{\partial z^k} \otimes \boldsymbol{g}^k \tag{32}$$

$$= \left(\frac{\partial \sigma^{ij}}{\partial z^k} + +\sigma^{lj}\Gamma^i_{lk} + \sigma^{il}\Gamma^j_{lk}\right)\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}^k\,. \tag{33}$$

The divergence is obtained by contracting upon the j and k indicies to give

$$\nabla \cdot \boldsymbol{\sigma} = \left(\frac{\partial \sigma^{ij}}{\partial z^j} + +\sigma^{lj}\Gamma^i_{lj} + \sigma^{il}\Gamma^j_{lj}\right)\boldsymbol{g}_i\,. \tag{34}$$

For our two coordinate systems we can expand this last expression and convert to physical components. The result in the physical basis for the cylindrical coordinate systems is:

$$\begin{aligned}
\boldsymbol{e}_r \cdot (\nabla \cdot \boldsymbol{\sigma}) &= \sigma_{rr,r} + \frac{1}{r}\sigma_{r\theta,\theta} + \sigma_{rz,z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} \\
\boldsymbol{e}_\theta \cdot (\nabla \cdot \boldsymbol{\sigma}) &= \sigma_{\theta r,r} + \frac{1}{r}\sigma_{\theta\theta,\theta} + \sigma_{\theta z,z} + \frac{2\sigma_{\theta r}}{r} \\
\boldsymbol{e}_z \cdot (\nabla \cdot \boldsymbol{\sigma}) &= \sigma_{zr,r} + \frac{1}{r}\sigma_{z\theta,\theta} + \sigma_{zz,z} + \frac{\sigma_{zr}}{r}
\end{aligned} \tag{35}$$

The result in the physical basis for the spherical coordinate systems is:

$$\begin{aligned}
\boldsymbol{e}_r \cdot (\nabla \cdot \boldsymbol{\sigma}) &= \sigma_{rr,r} + \frac{1}{r}\sigma_{r\varphi,\varphi} + \frac{1}{r\sin(\varphi)}\sigma_{r\theta,\theta} + \frac{2\sigma_{rr} - \sigma_{\varphi\varphi} - \sigma_{\theta\theta} + \sigma_{r\varphi}\cot(\varphi)}{r} \\
\boldsymbol{e}_\varphi \cdot (\nabla \cdot \boldsymbol{\sigma}) &= \sigma_{\varphi r,r} + \frac{1}{r}\sigma_{\varphi\varphi,\varphi} + \frac{1}{r\sin(\varphi)}\sigma_{\varphi\theta,\theta} + \frac{3\sigma_{\varphi r} + (\sigma_{\varphi\varphi} - \sigma_{\theta\theta})\cot(\varphi)}{r} \\
\boldsymbol{e}_\theta \cdot (\nabla \cdot \boldsymbol{\sigma}) &= \sigma_{\theta r,r} + \frac{1}{r}\sigma_{\theta\varphi,\varphi} + \frac{1}{r\sin(\varphi)}\sigma_{\theta\theta,\theta} + \frac{3\sigma_{r\theta} + 2\sigma_{\varphi\theta}\cot(\varphi)}{r}
\end{aligned} \tag{36}$$