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Compatibility – Cesaro's Integral

If a displacement field is to be well behaved (i.e. single valued) then the line integral around a closed curve of the form:

$$\oint \mathrm{d}u_i = 0\,. \tag{1}$$

This integral is Cesaro's integral. Rewrite $\mathrm{d}u_i = u_{i,j}\mathrm{d}x_j = (\varepsilon_{ij} + \omega_{ij})\mathrm{d}x_j$ so that

$$\oint (\varepsilon_{ij} + \omega_{ij})\mathrm{d}x_j = 0 \tag{2}$$

As an aside consider that

$$\oint \omega_{ij}\mathrm{d}x_j = \oint [(x_l\omega_{il})_{,j} - x_l\omega_{il,j}]\mathrm{d}x_j = -\oint x_l\omega_{il,j}\mathrm{d}x_j \tag{3}$$

since the integral of the first term yields zero when integrated around a closed curve. As a second aside consider that

$$\begin{aligned}\omega_{il,j} &= \tfrac{1}{2}(u_{i,lj} - u_{l,ij}) + \tfrac{1}{2}(u_{j,il} - u_{j,il}) \\ &= \tfrac{1}{2}(u_{i,lj} + u_{j,il}) - \tfrac{1}{2}(u_{l,ij} + u_{j,il}) \\ &= \varepsilon_{ij,l} - \varepsilon_{lj,i} = (\delta_{ip}\delta_{lq} - \delta_{lp}\delta_{iq})\varepsilon_{pj,q} \\ &= e_{mil}e_{mpq}\varepsilon_{pj,q}\,.\end{aligned} \tag{4}$$

Returning these results to (2) gives

$$\oint \mathrm{d}u_i = \oint [\varepsilon_{ij} - x_l e_{mil}e_{mpq}\varepsilon_{pj,q}]\mathrm{d}x_j\,. \tag{5}$$

Note now that $\oint \boldsymbol{F}\cdot d\boldsymbol{x} = \int\int_S \boldsymbol{n}\cdot\nabla\times\boldsymbol{F}\mathrm{d}S$ where S is the surface/area surrounded by the closed curve. Thus one may convert (5) to a surface/area integral

$$\begin{aligned}\oint \mathrm{d}u_i &= \int\int_S n_r e_{rsj}[\varepsilon_{ij} - x_l e_{mil}e_{mpq}\varepsilon_{pj,q}]_{,s}\mathrm{d}S \\ &= \int\int_S \underbrace{n_r e_{rsj}}_{(a)}[\underbrace{\varepsilon_{ij,s}}_{(b)} - \underbrace{\delta_{ls}e_{mil}e_{mpq}\varepsilon_{pj,q}}_{(c)} - x_l e_{mil}e_{mpq}\varepsilon_{pj,qs}]\mathrm{d}S\,.\end{aligned} \tag{6}$$

Term (c) can be split into two parts as

$$e_{mis}e_{mpq}\varepsilon_{pj,q} = (\delta_{ip}\delta_{sq} - \delta_{iq}\delta_{sp})\varepsilon_{pj,q} = \underbrace{\varepsilon_{ij,s}}_{(c1)} - \underbrace{\varepsilon_{sj,i}}_{(c2)}\,. \tag{7}$$

Note that term $(c1)$ cancels with (b) and (a) operating on $(c2)$ produces zero. Thus

$$\begin{aligned} \oint \mathrm{d}u_i &= -\int\int_S n_r e_{rsj} x_l e_{mil} e_{mpq} \varepsilon_{pj,qs} \mathrm{d}S \\ &= -\int\int_S n_r e_{mil}(e_{rsj}e_{mpq}\varepsilon_{pj,qs}) x_l \mathrm{d}S\,. \end{aligned} \tag{8}$$

The term in the parenthesis represents the curl of the curl of the small strain tensor. Thus a *sufficient condition* for a single values displacement field is that

$$\nabla \times \nabla \times \boldsymbol{\varepsilon} = 0 \rightarrow e_{rsj}e_{mpq}\varepsilon_{pj,qs} = 0\,. \tag{9}$$

With the result from lecture that showed that (9) was a necessary condition for single-valuedness, we have the final result that (9) is a necessary and sufficient condition for a single valued displacement field. In the presence of a multiply connected body (i.e. one with holes) there are several additional conditions that must hold. These can be found in many comprehensive books on elasticity.