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Cauchy's Theorem

Theorem 1 (Cauchy's Theorem) *Let $\boldsymbol{T}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ be a system of forces for a body Ω. Then a necessary and sufficient condition that the global momentum balance laws be satisfied is that there exist a spatial tensor field $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ called the Cauchy stress such that*

$$\text{(a)} \quad \text{For each } \boldsymbol{n} \text{ (unit) } \boldsymbol{T}(\boldsymbol{n}) = \boldsymbol{\sigma}^T \boldsymbol{n} \tag{1}$$

$$\text{(b)} \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{2}$$

$$\text{(c)} \quad \text{div}[\boldsymbol{\sigma}^T] + \boldsymbol{B} = \rho \ddot{\boldsymbol{u}} \tag{3}$$

[**Non-Rigorous Proof for necessity** sufficiency will be assigned for homework]. Note that necessity means if we assume global momentum balance then (a), (b), and (c) follows. Sufficiency means if we assume (a), (b), and (c) then global momentum balance follows.

The first step in the proof is to prove that $\boldsymbol{\sigma}$ exists and is a tensor. We begin by considering *Cauchy's Tetrahedron* and performing momentum balance upon it. Consider a tetrahedron that has been cut out of a deformed body as shown in Fig. 1. If the tetrahedron is small enough[1], then an application of global momentum balance yields:

$$\begin{aligned} &\boldsymbol{T}(-\boldsymbol{e}_1)\, \boldsymbol{e}_1 \cdot \boldsymbol{n} dA + \boldsymbol{T}(-\boldsymbol{e}_2)\, \boldsymbol{e}_2 \cdot \boldsymbol{n} dA + \boldsymbol{T}(-\boldsymbol{e}_3)\, \boldsymbol{e}_3 \cdot \boldsymbol{n} dA \\ &+ \boldsymbol{T}(\boldsymbol{n}) dA + B \frac{1}{3} h dA = \rho \dot{\boldsymbol{v}} \frac{1}{3} h dA \,. \end{aligned} \tag{4}$$

Now note that $\boldsymbol{T}(-\boldsymbol{e}_i) = -\boldsymbol{T}(\boldsymbol{e}_i)$. This can be seen by either appealing to Newton's Third Law or better by applying momentum balance individually to the faces of the tetrahedron. Thus have

$$\begin{aligned} &-\boldsymbol{T}(\boldsymbol{e}_1)\boldsymbol{e}_1 \cdot \boldsymbol{n} dA - \boldsymbol{T}(\boldsymbol{e}_2)\boldsymbol{e}_2 \cdot \boldsymbol{n} dA - \boldsymbol{T}(\boldsymbol{e}_3)\boldsymbol{e}_3 \cdot \boldsymbol{n} dA \\ &+ \boldsymbol{T}(\boldsymbol{n}) dA + B \frac{1}{3} h dA = \rho \dot{\boldsymbol{v}} \frac{1}{3} h dA \,. \end{aligned} \tag{5}$$

[1] Note that one can also appeal to the mean value theorem to make this step more precise.

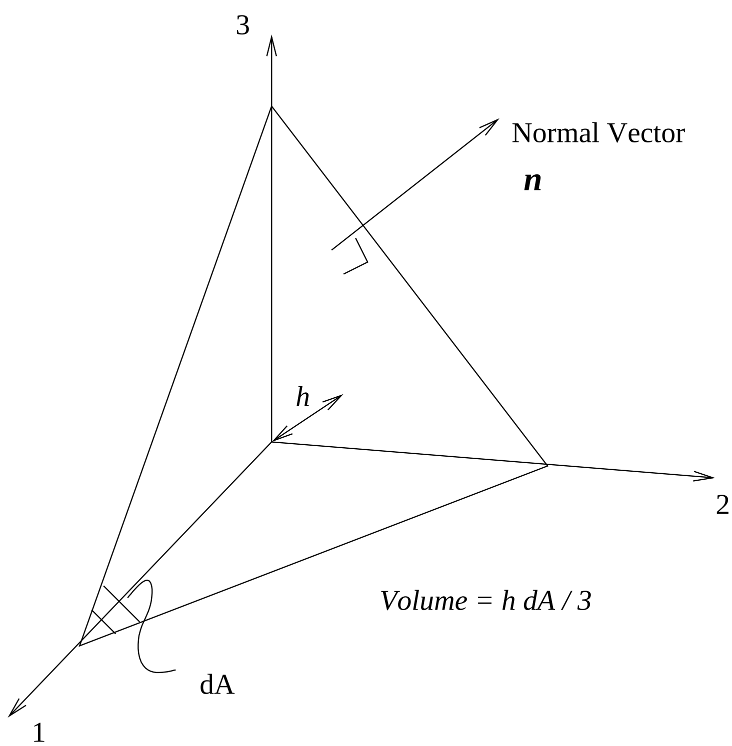

Figure 1: Cauchy's Tetrahedron.

Dividing through by the area dA and taking the limit as $h \to 0$ gives:

$$\boldsymbol{T}(\boldsymbol{e}_1)n_1 + \boldsymbol{T}(\boldsymbol{e}_2)n_2 + \boldsymbol{T}(\boldsymbol{e}_3)n_3 = \boldsymbol{T}(\boldsymbol{n}) = \boldsymbol{T}(n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3)\,. \tag{6}$$

Thus we see that the traction is infact a linear operator from $\mathbb{R}^3$ to $\mathbb{R}^3$. This means that we can describe the traction using a tensor. We will call this tensor the Cauchy stress tensor and use the symbol $\boldsymbol{\sigma}$ to denote it. Adhering to convention we will equate the action of the transpose of $\boldsymbol{\sigma}$ to the traction itself; thus,

$$\boldsymbol{T}(\boldsymbol{n}) = \boldsymbol{\sigma}^T\boldsymbol{n}\,. \tag{7}$$

Using this relation we can also make clear interpretations of the meaning of the components of the stress tensor. For example, if we let $\boldsymbol{n} = \boldsymbol{e}_i$, then we find that

$$\boldsymbol{e}_j \cdot \boldsymbol{T}(\boldsymbol{e}_i) = \sigma_{ij}\,. \tag{8}$$

Thus we see that σ_{ij} represents traction components (force per unit area) in the $\boldsymbol{e}_j$ direction on a section cut with normal vector $\boldsymbol{e}_i$.

Necessity of (c): From linear momentum balance for an arbitrary part $\mathcal{P} \subset \Omega$ with boundary $\partial\mathcal{P}$

$$\int_{\partial\mathcal{P}} \boldsymbol{T}\,dA + \int_{\mathcal{P}} \boldsymbol{B}\,dV = \int_{\mathcal{P}} \rho\ddot{\boldsymbol{u}}\,dV\,. \tag{9}$$

By (a) the first term can be written as

$$\int_{\partial\mathcal{P}} \boldsymbol{T}\,dA = \int_{\partial\mathcal{P}} \boldsymbol{\sigma}^T\boldsymbol{n}\,dA = \int_{\mathcal{P}} \mathrm{div}[\boldsymbol{\sigma}^T]\,dV\,. \tag{10}$$

If we plug this back in (9), then

$$\int_{\mathcal{P}} [\mathrm{div}[\boldsymbol{\sigma}^T] + \boldsymbol{B} - \rho\ddot{\boldsymbol{u}}]\,dV = 0\,. \tag{11}$$

Since this holds for any part $\mathcal{P}$ the integrand must equal zero; i.e. $\mathrm{div}[\boldsymbol{\sigma}^T] + \boldsymbol{B} = \rho\ddot{\boldsymbol{u}}$.

Necessity of (b): From angular momentum balance for an arbitrary part $\mathcal{P} \subset \Omega$ with boundary $\partial\mathcal{P}$

$$\int_{\partial\mathcal{P}} e_{ijk}x_jT_k\,dA + \int_{\mathcal{P}} e_{ijk}x_jB_k\,dV = \int_{\mathcal{P}} \rho e_{ijk}x_j\ddot{u}_k\,dV\,. \tag{12}$$

Begin by applying (a) to the first term so that

$$\int_{\partial\mathcal{P}} e_{ijk}x_jT_k\,dA = \int_{\partial\mathcal{P}} e_{ijk}x_j\sigma_{lk}n_l\,dA = \int_{\mathcal{P}} (e_{ijk}x_j\sigma_{lk})_{,l}\,dV \tag{13}$$

$$= \int_{\mathcal{P}} e_{ijk}x_{j,l}\sigma_{lk} + e_{ijk}x_j\sigma_{lk,l}\,dV \tag{14}$$

$$= \int_{\mathcal{P}} e_{ilk}\sigma_{lk} + e_{ijk}x_j\sigma_{lk,l}\,dV\,. \tag{15}$$

Plugging this result back into (12) gives

$$\int_{\mathcal{P}} e_{ilk}\sigma_{lk} + e_{ijk}x_j\underbrace{[\sigma_{lk,l} + B_k - \rho\ddot{u}_k]}_{=0 \text{ by (c)}}\,dV = 0\,. \tag{16}$$

Since this holds for any part $\mathcal{P}$ we have that $e_{ilk}\sigma_{lk} = 0$ and this in turn implies that

$$\sigma_{ij} = \sigma_{ji}\,. \tag{17}$$