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Linearization of Angle Change Relation

The general expression for the change of angle between two vectors $\boldsymbol{V}^{(1)}$ and $\boldsymbol{V}^{(2)}$ is given by:

$$\cos(\Theta) - \cos(\theta) = \boldsymbol{N}^{(1)} \cdot \left[\frac{\|\boldsymbol{V}^{(1)}\| \, \|\boldsymbol{V}^{(2)}\|}{\|\boldsymbol{F}\boldsymbol{V}^{(1)}\| \, \|\boldsymbol{F}\boldsymbol{V}^{(2)}\|} \boldsymbol{C} - \mathbf{1} \right] \boldsymbol{N}^{(2)} \,, \tag{1}$$

where $\|\boldsymbol{V}^{(i)}\| << 1$ and $\boldsymbol{N}^{(i)} = \boldsymbol{V}^{(i)}/\|\boldsymbol{V}^{(i)}\|$. The linearization of both sides gives:

$$-\sin(\Theta)\Delta\theta = -(\boldsymbol{N}^{(1)} \cdot \boldsymbol{N}^{(2)})(\boldsymbol{N}^{(1)} \cdot \boldsymbol{\varepsilon} \boldsymbol{N}^{(1)} + \boldsymbol{N}^{(2)} \cdot \boldsymbol{\varepsilon} \boldsymbol{N}^{(2)}) + 2\boldsymbol{N}^{(1)} \cdot \boldsymbol{\varepsilon} \boldsymbol{N}^{(2)} \,, \tag{2}$$

where $\theta = \Theta + \Delta\theta$ and $|\Delta\theta| << 1$. The expression above comes from linearizing Eq. (1). Note that the linearization of $\boldsymbol{C}$ about zero deformation gives:

$$\boldsymbol{C} \approx \mathbf{1} + \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T = \mathbf{1} + 2\boldsymbol{\varepsilon} \,. \tag{3}$$

Further note that the linearization about zero deformation of the term multiplying $\boldsymbol{C}$ is given by

$$\frac{\|\boldsymbol{V}^{(1)}\| \, \|\boldsymbol{V}^{(2)}\|}{\|\boldsymbol{F}\boldsymbol{V}^{(1)}\| \, \|\boldsymbol{F}\boldsymbol{V}^{(2)}\|} \approx 1 - \boldsymbol{N}^{(1)} \cdot \boldsymbol{\varepsilon} \boldsymbol{N}^{(1)} - \boldsymbol{N}^{(2)} \cdot \boldsymbol{\varepsilon} \boldsymbol{N}^{(2)} \,. \tag{4}$$

This last expression can be found by noting that to linear approximation: (a) $\|\boldsymbol{F}\boldsymbol{V}^{(i)}\| \approx \|\boldsymbol{V}^{(i)}\| \, (1 + \boldsymbol{N}^{(i)} \cdot \boldsymbol{\varepsilon} \boldsymbol{N}^{(i)})$ and (b) $\frac{1}{1+x} \approx 1 - x$.

The test of the limit case of $\boldsymbol{N}^{(2)} \to \boldsymbol{N}^{(1)}$ in Eq. (2) produces the result of

$$0 \cdot \Delta\theta = 0 \,, \tag{5}$$

which is an indeterminate result – though still completely consistent. If we use the LHS of Eq. (1) with the RHS of Eq. (2), instead, we find that

$$\cos(\Theta) - \cos(\Theta + \Delta\theta) = 0 \,, \tag{6}$$

which implies $\Delta\theta = 0$.