UNIVERSITY OF CALIFORNIA BERKELEY Department of Civil Engineering Fall 2016

Structural Engineering, Mechanics and Materials Professor: S. Govindjee

**Michell's Solution to the Bi-Harmonic**

Michell's solution to the Bi-Harmonic equation is a series solution in polar coordinates. In polar coordinates the bi-harmonic equation reads

$$\nabla^4\Phi = \left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\theta^2}\right)\left(\Phi_{,rr} + \frac{1}{r}\Phi_{,r} + \frac{1}{r^2}\Phi_{,\theta\theta}\right) = 0\,. \tag{1}$$

In this coordinate chart we also have the relations

$$\sigma_{rr} = r^{-1}\Phi_{,r} + r^{-2}\Phi_{,\theta\theta} \tag{2}$$

$$\sigma_{\theta\theta} = \Phi_{,rr} \tag{3}$$

$$\sigma_{r\theta} = -\left(r^{-1}\Phi_{,\theta}\right)_{,r}\,. \tag{4}$$

The Michell solution to $\nabla^4\Phi = 0$ is constructed in two parts. The first part is given by considering a separable solution in the spirit of Levy as $f_n(r)\exp[\pm in\theta]$. This will give all the solutions to the bi-harmonic that are single-valued and $2\pi$ periodic. Plugging into the bi-harmonic equation results in an ordinary differential equation for $f_n(r)$. This equation has constant coefficients and can be easily solved (though, due to repeated roots there is a need to employ "variation of parameters" ). It needs to be noted that what actually needs to be single-valued and $2\pi$ periodic are the stresses and that there are actually solutions of the form $f_m(r)g_m(\theta)$ where $g_m(\theta)$ are not sines and cosines but where the stresses are still single-valued. While not a complete enumeration, the standard solution looks for solutions of the form $r^m g_m(\theta)$, for $m \in \{0,1,2\}$. The final result is a nearly complete series solution to the bi-harmonic equation:

$$\begin{aligned}
\Phi &= a_o\ln(r) + b_o r^2 + c_o r^2\ln(r) + d_o r^2\theta + a_o'\theta && (5)\\
&+ a_1 r\theta\sin(\theta) + \left(b_1 r^3 + a_1'/r + b_1' r\ln(r)\right)\cos(\theta) && (6)\\
&+ c_1 r\theta\cos(\theta) + \left(d_1 r^3 + c_1'/r + d_1' r\ln(r)\right)\sin(\theta) && (7)\\
&+ \sum_{n=2}^{\infty}\left(a_n r^n + b_n r^{n+2} + a_n'/r^n + b_n'/r^{n-2}\right)\cos(n\theta) && (8)\\
&+ \sum_{n=2}^{\infty}\left(c_n r^n + d_n r^{n+2} + c_n'/r^n + d_n'/r^{n-2}\right)\sin(n\theta) && (9)
\end{aligned}$$

where $a_i$ *etc.* are constants.

**Remarks:**

1. Terms in the solution that result in identically zero stresses have been left out.

2. A number of the terms in the solution lead to singularities at the origin and thus need to be zero if the origin of the coordinate system is a material point (in the absence of point force at the origin).

3. If the body is multiply connected, then to ensure single-valued solutions the following integrability conditions on each internal cavity must be satisfied:

$$\int_{C_i} \frac{d}{dn}(\nabla^2\Phi)\, ds = 0 \tag{10}$$

$$\int_{C_i} \left[ y\frac{d}{ds}(\nabla^2\Phi) + x\frac{d}{dn}(\nabla^2\Phi) \right] ds = -\frac{1}{1-\nu}B_x \tag{11}$$

$$\int_{C_i} \left[ y\frac{d}{dn}(\nabla^2\Phi) - x\frac{d}{ds}(\nabla^2\Phi) \right] ds = -\frac{1}{1-\nu}B_y \tag{12}$$

   where $d/ds$ and $d/dn$ respectively denote tangential and normal derivatives to the cavity boundary and $B_x$ and $B_y$ are the body force components. The first of these conditions requires $a_o = 0$ if the origin is surrounded by an internal boundary.

4. The solution above comes from a paper by J.H. Michell, *Proc. London Math Soc.*, vol. 31, p.100, 1899. A disscusion on the use of this solution can be found in *Theory of Elasticity* by S.P. Timoshenko and J.N. Goodier, Art. 43-46. A more comprehensive discussion of the properties of the solution to the bi-harmonic equation in the context of elasticity may be found in *Mathematical Theory of Elasticity* by I.S. Sokolnikoff, Art. 69-70. The essense of the art of using the general solution is to understand through the computation of examples what the individual terms do. Then to consider their linear combinations in such a way as to solve practical problems of interest.

5. Note that at times the Michell solution can be lacking. For instance, while expressions of the form $\theta \ln(r)$ and $r^2\theta \ln(r)$ are solutions to the bi-harmonic equation, they are not part of Michell' solutions as they

are neither of the form $f_n(r)\exp[\pm in\theta]$ nor of the form $r^m g_m(\theta)$; note $\ln(r)$ does not even possess a Laurent series expansion. So while the solution includes virtually all cases of interest, it does not contain all solutions. Thus if a problem can not be solved using Michell's solution, then one may need to consider other solution forms, for example, $\theta f(r)$ among others.

6. The wikipedia page http://en.wikipedia.org/wiki/Michell_solution conveniently lists the resulting expressions for the stress components and displacement components for each term in the above expansion. Note that each term by itself satisfies the bi-harmonic equation. Thus by looking through the listing on the website you can find the terms you need by mixing and matching to the boundary conditions you have in your problem.