UNIVERSITY OF CALIFORNIA BERKELEY
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**Steady State Analysis and Exponential Notation**

# 1 Linear Ordinary Differential Equations

In the solution of equations describing periodically forced dynamical systems one is often only interested in the steady state solution – i.e. the solution after all transients have died out. When the system is governed by a linear differential equation, we can take advantage of Euler's formula $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$ to find just the forced part of the solution; here and throughout $i = \sqrt{-1}$. Using Euler's formula is typically much easier than employing direct methods for the solution of ordinary differential equations.

As an example let us consider the following simple dynamical system

$$\ddot{x} + \dot{x} + x = f(t)\,, \tag{1}$$

where we wish to solve for $x(t)$ subject to the initial conditions $x(0) = x_o$ and $\dot{x}(0) = v_o$. The load $f(t) = F\sin(\omega t)$, where $\omega$ and $F$ are given constants.

The classical method of solution requires one to first find the homogenous solution (i.e. the solution where $f(t) = 0$). This is usually done using an assumed form $x = Ae^{st}$, solving the characteristic polynomial for $s$, and then recombining the terms. The result of this process is

$$x_H(t) = e^{-t/2}\left[A\cos\left(\frac{\sqrt{3}}{2}t\right) + B\sin\left(\frac{\sqrt{3}}{2}t\right)\right]\,, \tag{2}$$

where $A$ and $B$ are constants that still have to be determined. It then remains to find a particular solution $x_P(t)$. This is usually accomplished by educated guess work and trial-and-error. For our problem the result is

$$x_P(t) = F\left[\frac{1-\omega^2}{1-\omega^2+\omega^4}\sin(\omega t) - \frac{\omega}{1-\omega^2+\omega^4}\cos(\omega t)\right]\,. \tag{3}$$

The total solution $x(t) = x_H(t) + x_P(t)$. The two constants $A$ and $B$ are then found by requiring $x(0) = x_o$ and $\dot{x}(0) = v_o$.

If we are only interested in the forced part of the solution (the particular solution), then we can employ complex variable notation and make the process algebraically easier. Assume now that $f(t) = Fe^{i\omega t}$. In this case, for all linear systems, the particular solution $x_P(t) = X_P e^{i\omega t}$; the only unknown in this functional form is $X_P$ which happens to be a complex number in general. The physical interpretation is that the real part of $X_P$ provides the magnitude of the forced response in phase with the load and the imaginary part is the magnitude of the out of phase (by 90 degrees) part of the solution. For our problem, if we insert these two forms for $x_P$ and $f$, then we find

$$(-\omega^2 X_P + i\omega X_P + X_P)e^{i\omega t} = Fe^{i\omega t}\,. \tag{4}$$

Thus

$$X_P = \frac{F}{-\omega^2 + i\omega + 1} = \underbrace{F\frac{1-\omega^2}{1-\omega^2+\omega^4}}_{\text{in phase part}} + i\,\underbrace{F\frac{-\omega}{1-\omega^2+\omega^4}}_{\text{out of phase part}}\,. \tag{5}$$

The interpretation of this result is that if the real input was $f(t) = F\sin(\omega t)$, then the force output will be exactly what is shown in Eq. 3. Note that if the input were instead $f(t) = \sin(\omega t + \delta)$, then the same interpretation holds but one has to properly understand the notions of *in phase* and *out of phase*. For this case we would have

$$x_P(t) = F\left[\frac{1-\omega^2}{1-\omega^2+\omega^4}\sin(\omega t+\delta) - \frac{\omega}{1-\omega^2+\omega^4}\cos(\omega t+\delta)\right]\,. \tag{6}$$

Further note that if the input were instead $f(t) = cos(\omega t)$, then the output will be the real part of our complex result:

$$x_P(t) = \mathrm{Re}[X_P e^{i\omega t}] = F\left[\frac{1-\omega^2}{1-\omega^2+\omega^4}\cos(\omega t) + \frac{\omega}{1-\omega^2+\omega^4}\sin(\omega t)\right] \tag{7}$$

From Eq. 5, one can also make the following two useful interpretations:

1. The magnitude/modulus of $X_P$, i.e. $|X_P| = \sqrt{\mathrm{Re}(X_P)^2 + \mathrm{Im}(X_P)^2}$, provides the amplitude of the forced response.

2. The angle of $X_P$, i.e. $\angle X_P = \tan^{-1}(\mathrm{Im}(X_P)/\mathrm{Re}(X_P))$, provides the phase angle shift of the system's forced response relative to the applied forcing function.

Thus an alternate, and very useful way of writing the forced solutions, is

$$x_P(t) = |X_P|e^{i(\omega t + \angle X_P)}\,. \tag{8}$$

A classical way of visualizing these solutions is to construct the so-called Bode plots of the transfer function; i.e. the graphs of $|X_P/F|$ and $\angle X_P$ versus $\log_{10}(\omega)$. For our example equation, these are shown in Fig. 1. As the top of Fig. 1 shows, at low forcing frequencies the response amplitude to input amplitude ratio is unity, at a frequency of $\omega = 1/\sqrt{2}$ rad/s it peaks, and then at higher forced frequencies the system ceases to respond. For each frequency the response is phase shifted with respect to the input as shown in the bottom of Fig. 1. At low frequencies there is almost no phase shift; as the loading frequency is increased the response lags behind the load; at $\omega = 1$ rad/s the response is a 90 degrees out of phase with the load (i.e. at zero load the response is at its maximal value), then finally at high frequencies it is out of phase by a full 180 degrees (i.e. maximal positive response at maximal negative load ).

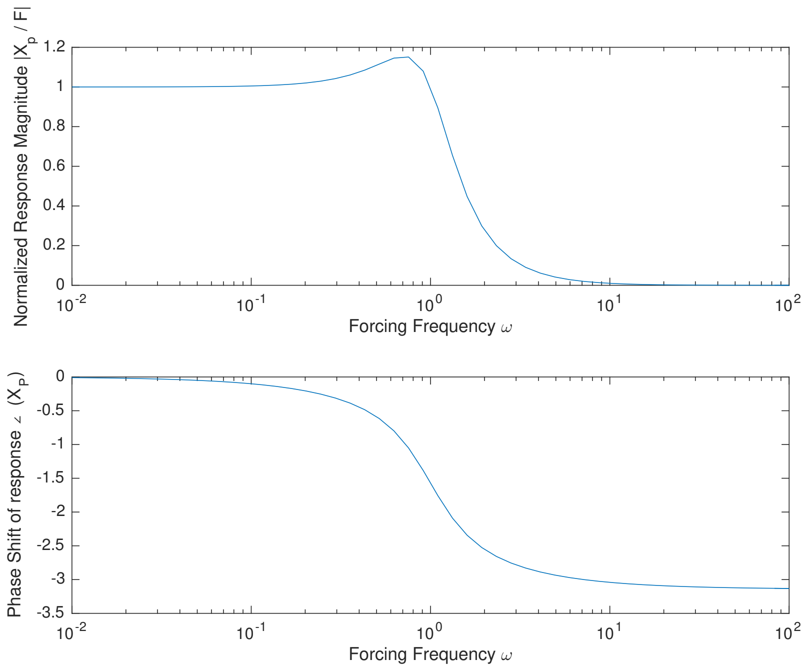


Figure 1: Bode plots for the example problem

# 2 General Mathematical Idea

The utility of using complex notation carries over to any linear relation. Suppose a real-valued function $f(t)$ is related to a real-valued function $x(t)$ via a relation of the form

$$f(t) = L[x(t)]\,, \tag{9}$$

where $L$ is any linear operator; in particular it could be a differential operator or an integral operator. If it is a differential operator one would generally think of $f(t)$ as the input and $x(t)$ as the output; in the integral operator case this nomenclature would generally be reversed. Either way it important to observe that in actual physical systems these inputs and outputs will be real (in the sense of complex numbers). Notwithstanding, it is generally useful to think of them as complex quantities. In that case

$$\begin{aligned} f_R(t) + i f_I(t) &= L[x_R(t) + i x_I(t)] & (10)\\ &= L[x_R(t)] + L[i x_I(t)] & (11)\\ &= L[x_R(t)] + i\, L[x_I(t)] & (12) \end{aligned}$$

where linearity of $L$ has been exploited. Taking the real and imaginary parts of Eq. 12, gives us the result that

$$f_R(t) = L[x_R(t)] \qquad \text{and} \qquad f_I(t) = L[x_I(t)]\,. \tag{13}$$

It is for this very basic reason we can replace an oscillating input with $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ and then solve for the output, all in complex form. Then finally take either the real part or the imaginary part for the final answer, depending on if the actual input was either a sine or a cosine.