Estimation of the Fundamental Mode of a Radial-Disk Resonator

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Abstract

In this brief note we present a method of estimating the first resonant mode of a radial-disk resonator. The methodology employed utilizes a classical Ritz approximation to the relevant eigenvalue problem. With a single term approximation a very accurate approximation can be computed.

1 Introduction

Let us consider the estimation of the first radial vibration mode of a circular disk. The geometry is shown in Fig. 1. It consists of a homogeneous isotropic linear elastic circular disk of radius R and unspecified thickness; we will consider the two extremal cases of zero thickness strain and zero thickness stress, viz. $\varepsilon_{zz} = 0$ and $\sigma_{zz} = 0$, respectively.

2 Governing Equations

The relevant kinematic relations for the geometry considered are

$$\varepsilon_{rr} = u_{,r} \tag{1}$$

$$\varepsilon_{\theta\theta} = u/r\,. \tag{2}$$

In the thickness direction we will either assume $\varepsilon_{zz} = 0$ or determine it from the zero thickness stress condition. Note, $u(r,t)$ is the radial motion of the

Figure 1: Disk geometry

material. Within the chosen approximations it is only a function of radial postition r and time t.

In this setting there is only one non-trivial equilibrium equation:

$$\sigma_{rr,r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \ddot{u}\,, \tag{3}$$

where ρ is the material density.

In the zero thickness strain case, the constitutive response is given as

$$\sigma_{rr} = (2\mu + \lambda)\varepsilon_{rr} + \lambda\varepsilon_{\theta\theta} \tag{4}$$
$$\sigma_{\theta\theta} = (2\mu + \lambda)\varepsilon_{\theta\theta} + \lambda\varepsilon_{rr}\,, \tag{5}$$

where μ is the shear modulus and λ the Lamé modulus. For the zero thickness stress case, the constitutive response is given as

$$\sigma_{rr} = \left(2\mu + \lambda - \frac{\lambda^2}{2\mu + \lambda}\right)\varepsilon_{rr} + \left(\lambda - \frac{\lambda^2}{2\mu + \lambda}\right)\varepsilon_{\theta\theta} \tag{6}$$
$$\sigma_{\theta\theta} = \left(2\mu + \lambda - \frac{\lambda^2}{2\mu + \lambda}\right)\varepsilon_{\theta\theta} + \left(\lambda - \frac{\lambda^2}{2\mu + \lambda}\right)\varepsilon_{rr}\,. \tag{7}$$

In what follows, we will write all expressions for the zero thickness strain case. To convert to the zero thickness stress case, one only needs to replace all instances of λ by $2\mu\lambda/(2\mu + \lambda)$.

Combining the above relations, one can derive a Navier-form of the governing relation:

$$u_{,rr} + (u/r)_{,r} = \frac{\rho}{2\mu + \lambda}\ddot{u}\,. \tag{8}$$

This relation needs to be solved in the steady state subject to the boundary condtions: $u(0,t) = 0$ and $\sigma_{rr}(R,t) = 0$. To this end, one can assume a decomposition of $u(r,t) = f(r)\exp[i\omega t]$. Introducing this into (8) yield the eigen-problem:

$$f_{,rr} + (f/r)_{,r} = -\frac{\rho}{2\mu+\lambda}\omega^2 f\,, \tag{9}$$

where $f(0) = 0$ and $(2\mu+\lambda)f_{,r}(R) + \lambda f(R)/R = 0$.

3 Weak Formulation

The weak form of (9) reads:

$$\delta f(R)\frac{\lambda}{2\mu+\lambda}f(R) + \int_0^R \delta f_{,r} r f_{,r}\,dr + \int_0^R \delta f \frac{1}{r} f\,dr = \frac{\omega^2}{c^2}\int_0^R \delta f r f\,dr\,, \tag{10}$$

where we have assumed that the test function δf is zero at $r = 0$ and introduced the notation $c = \sqrt{(2\mu+\lambda)/\rho}$. Note that the stress boundary condition has already been incorporated into this expression.

4 Ritz Approximation

To estimate the first eigenvalue of our disk we can compute a Ritz estimate using the Galerkin approximation of $f(r) = \delta f(r) = -r^2 + 2Rr$. Plugging into both sides of (10) and solving for ω gives a circular frequency of:

$$\omega = \frac{c}{R}\sqrt{\frac{30}{11}\left(1.25 + \frac{\lambda}{2\mu+\lambda}\right)}\,. \tag{11}$$

As an example application consider the radial disk resonator in [1]. In this case, the elastic properties are given by $E = 139$ GPa and $\nu = 0.28$. The disk radius is $R = 41.5$ μm and the material density is given as $\rho = 4127$ kg/m^3. These parameters give for zero thickness strain an $\omega = 53.09$ MHz – compare to a resolved finite element computation which gives 52.9 MHz. In the case of zero thickness stress, these parameters give an $\omega = 47.36$ MHz, which compares well to the reported experimental value of 47.26 MHz. It should be noted that (11) corrects Equation (70) in [1] to read:

$$\omega = 2.04\frac{c}{R}\,, \tag{12}$$

where $c = 6045$ m/s is the zero thickness stress wave speed.

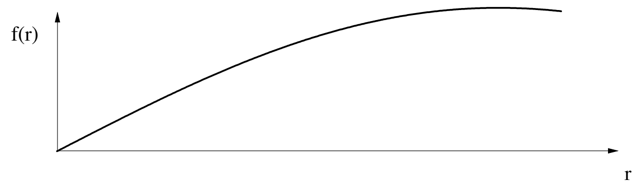

Figure 2: Radial mode shape from a FEA computation using FEAP [2] for the zero thickness strain case. The radial motion is plotted vertically.

It should be noted that the Ritz mode shape, while reasonable, does not respect the stress free end condition. A finite element computation of this same problem shows that our guess for the mode shape misses a small variation at the edge of the disk; see Fig. 2. Even as such, our simple quadratic approximation yields rather good results.

References

[1] D.S. Bindel and S. Govindjee. Elastic PMLs for resonator anchor loss simulation. *International Journal for Numerical Methods in Engineering*, 64:789–818, 2005.

[2] R.L. Taylor. FEAP. `http://www.ce.berkeley.edu/feap`, 2011.

A FEAP Inputs file for the generation of Fig. 2

```
feap  ** 1D radial disk resonator model zero thickness strain **
0 0 0 1 1 2

param
R  = 41.5d-6 ! disk radius
E  = 139d9   ! Young's Modulus
```

```
nu = 0.28    ! Poisson's Ratio
d  = 4127    ! Density
n  = 400     ! number of elements

block
cart n
1 0
2 R

eboun
1 0 1    ! Fix the inner radius

mate
solid
 axissymmetric
 elastic isotropic E nu
 density material  d


end


batch
 mass               ! Form mass
 tang               ! Form stiffness
 subs,,4            ! Compute eigenvalues
 plot,dofs,0,1      ! Map dof 1 to dof 2 for plotting
 plot,defo,,,1d-5   ! Rescale for plotting
 plot,eigv,1        ! Plot first eigenmode
end


inte
stop
```