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Anti-Plane Tearing and Mode III fault rupture

The tearing of a sheet of paper and the rupture of dip-slip faults can both be characterized as anti-plane strain problems. Consider the paper tearing case first. Assume a coordinate system (x_1, x_2) in the plane of the paper and x_3 out of plane. Then the process of tearing involves (approximately) a motion wherein $u_3(x_1, x_2)$ is the only non-zero component of the displacement field. In the case of fault rupture, if we align an (x_1, x_2) coordinate system such that x_2 is orthogonal to the fault plane, x_1 is in the fault plane and oriented towards the fault boundary, and x_3 is in the fault plane and oriented in the direction of uplift, then we also have a condition of anti-plane strain where $u_3(x_1, x_2)$ is the only non=zero component of displacement; see Fig. 1.

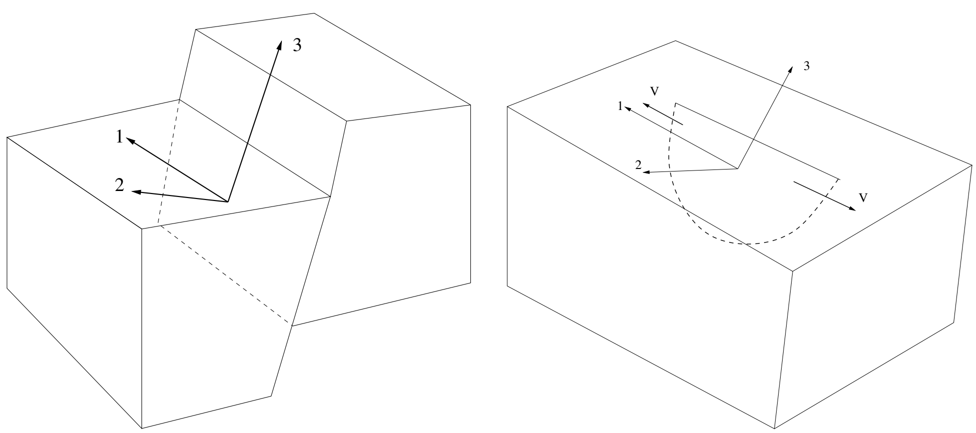

Figure 1: Mode III fault rupture diagram; zoom in (right), zoom out showing direction that the rupture is moving (left).

1 Anti-plane strain assumptions

Let us determine the stress field near the edge just where the material is tearing/rupturing. Looking in at the x_1, x_2 plane we will orient our coordinates as shown in Fig. 2.

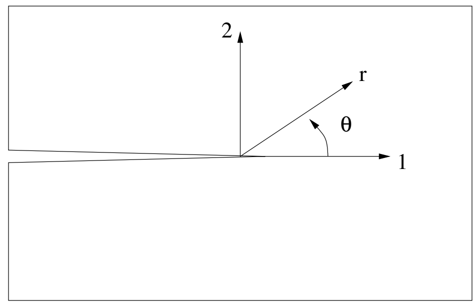

Figure 2: Mode III coordinate frame.

Assuming

$$u_1 = u_2 = 0 \qquad u_3(x_1, x_2)\,, \tag{1}$$

leads to only two non-zero strains

$$\varepsilon_{13} = \frac{1}{2}u_{3,1} \qquad \text{and} \qquad \varepsilon_{23} = \frac{1}{2}u_{3,2}\,. \tag{2}$$

Further assuming isotropic linear elastic behavior gives only two non-zero stresses:

$$\sigma_{13} = \mu u_{3,1} \qquad \text{and} \qquad \sigma_{23} = \mu u_{3,2}\,. \tag{3}$$

Plugging into the equilibrium equations, one finds that the Navier equation for equilibrium reduces to:

$$\nabla^2 u_3(x_1, x_2) = 0\,, \tag{4}$$

where we have assumed zero body forces, $\boldsymbol{b} = \mathbf{0}$. The overall problem will be to solve (4) subject to appropriate boundary conditions. Since our interest is in the stresses near the crack tip. We will assume that at some fixed radius $r = R$ that the tractions or displacements are given as boundary conditions. These will be supplemented by the observation that the crack faces are essentially traction free.

2 Conditions for stress-free crack faces

Let us first convert the traction free boundary conditions on $\theta = \pm\pi$ to expressions in terms of the displacements. This will be convenient since we will solve directly for the displacements from (4) and in polar form.

2.1 Stresses in terms of displacements

The two non-zero stresses can be written as

$$\begin{aligned}\sigma_{13} &= \mu u_{3,1} && (5)\\ &= \mu\left[\frac{\partial u_3}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial u_3}{\partial \theta}\frac{\partial \theta}{\partial x_1}\right] && (6)\\ &= \mu\left[\frac{\partial u_3}{\partial r}\cos(\theta) - \frac{\partial u_3}{\partial \theta}\frac{\sin(\theta)}{r}\right] && (7)\end{aligned}$$

and

$$\begin{aligned}\sigma_{23} &= \mu u_{3,2} && (8)\\ &= \mu\left[\frac{\partial u_3}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial u_3}{\partial \theta}\frac{\partial \theta}{\partial x_2}\right] && (9)\\ &= \mu\left[\frac{\partial u_3}{\partial r}\sin(\theta) + \frac{\partial u_3}{\partial \theta}\frac{\cos(\theta)}{r}\right]\,. && (10)\end{aligned}$$

The stress can be written as

$$\boldsymbol{\sigma} = \sigma_{13}(\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_1) + \sigma_{23}(\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_2)\,. \tag{11}$$

Thus the polar stresses, using (7) and (10), can be expressed as

$$\begin{aligned}\sigma_{r3} &= \boldsymbol{e}_r \cdot \boldsymbol{\sigma}\boldsymbol{e}_3 = \sigma_{13}(\boldsymbol{e}_r \cdot \boldsymbol{e}_1) + \sigma_{23}(\boldsymbol{e}_r \cdot \boldsymbol{e}_2) && (12)\\ &= \sigma_{13}\cos(\theta) + \sigma_{23}\sin(\theta) = \mu u_{3,r} && (13)\end{aligned}$$

and

$$\begin{aligned}\sigma_{\theta 3} &= \boldsymbol{e}_\theta \cdot \boldsymbol{\sigma}\boldsymbol{e}_3 && (14)\\ &= -\sigma_{13}\sin(\theta) + \sigma_{23}\cos(\theta) = \mu u_{3,\theta}/r\,. && (15)\end{aligned}$$

2.2 Zero traction boundary condition

The zero traction boundary condition on the crack faces says that

$$\boldsymbol{\sigma}(r,\pi)\boldsymbol{e}_\theta(\pi) = \boldsymbol{0} \tag{16}$$

on the top face and that

$$\boldsymbol{\sigma}(r,-\pi)\boldsymbol{e}_\theta(-\pi) = \boldsymbol{0} \tag{17}$$

on the bottom crack face. Noting that one can express the stress as

$$\boldsymbol{\sigma} = \sigma_{r3}(\boldsymbol{e}_r \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_r) + \sigma_{\theta 3}(\boldsymbol{e}_\theta \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_\theta) \tag{18}$$

allows us to express the sterss acting on surface normal $\boldsymbol{e}_\theta$ as $\boldsymbol{\sigma}\boldsymbol{e}_\theta = \sigma_{3\theta}\boldsymbol{e}_3$. This in turn implies that $\sigma_{3\theta}(r, \pm\pi) = 0$ and subsequently that

$$u_{3,\theta}(r, \pm\pi) = 0 \tag{19}$$

as the crack face boundary condition in terms of the displacements.

3 Computing the displacement field

To determine the displacement field we need to solve (4). We will do so in polar coordinates. To begin let us assume a separable solution of the form $u_3 = r^\lambda f(\theta)$, where the scalar λ and the function $f(\cdot)$ are both unknown. Plugging into (4) gives

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) r^\lambda f(\theta) = 0 \tag{20}$$

$$\lambda(\lambda - 1)r^{\lambda-2} f + \lambda r^{\lambda-2} f + r^{\lambda-2} f'' = 0 \tag{21}$$

$$\lambda^2 f + f'' = 0\,. \tag{22}$$

The solution to (22) (for $\lambda \neq 0$) is

$$f = A\cos(\lambda\theta) + B\sin(\lambda\theta)\,. \tag{23}$$

Thus the solution to the displacement field is of the form

$$u_3 = r^\lambda \left(A\cos(\lambda\theta) + B\sin(\lambda\theta)\right)\,. \tag{24}$$

Should $\lambda = 0$, then the solution is $u_3 = A + B\theta$; i.e. $f(\theta)$ is a linear function and $r^0 = 1$. We omit the constant term as it represents a rigid body motion (and hence generates no stresses), and we omit the linear term as it leads to non-zero shear strains of the form $\varepsilon_{\theta 3} = Br^{-1}$. Such strains can not be present as they correspond to a strain energy density near the crack tip that is $O(r^{-2})$, which when integrated over a small volume near the crack tip gives an infinite energy. The unknown parameter $\lambda \neq 0$ can be determined from the boundary conditions on the crack faces:

$$r^\lambda \left(-\lambda A\sin(\lambda\pi) + \lambda B\cos(\lambda\pi)\right) = 0 \tag{25}$$

$$r^\lambda \left(-\lambda A\sin(-\lambda\pi) + \lambda B\cos(-\lambda\pi)\right) = 0\,. \tag{26}$$

For a non-trivial solution (for A and B) we need that

$$\det \begin{bmatrix} -\sin(\lambda\pi) & \cos(\lambda\pi) \\ \sin(\lambda\pi) & \cos(\lambda\pi) \end{bmatrix} = 0 \tag{27}$$

Thus $-2\cos(\lambda\pi)\sin(\lambda\pi) = -\sin(\lambda 2\pi) = 0$. Hence,

$$\lambda_n = \frac{n}{2}\,, \tag{28}$$

where $n = \cdots, -2, -1, 1, 2, \cdots$ – i.e. any integer (but zero). The complete displacement solution is thus given as

$$u_3(r,\theta) = \sum_{n\neq 0} r^{n/2}\left[A_n \cos\left(\frac{n\theta}{2}\right) + B_n \sin\left(\frac{n\theta}{2}\right)\right]. \tag{29}$$

Remarks:

1. Negative values of n lead to *non-physical* infinite displacements at the crack tip. Thus negative values should be omitted, and the displacement field is actually of the form:

$$u_3(r,\theta) = \sum_{n=1}^{\infty} r^{n/2}\left[A_n \cos\left(\frac{n\theta}{2}\right) + B_n \sin\left(\frac{n\theta}{2}\right)\right]. \tag{30}$$

   The stresses are given as

$$\sigma_{r3} = \mu \sum_{n=1}^{\infty} \frac{n}{2} r^{n/2-1}\left[A_n \cos\left(\frac{n\theta}{2}\right) + B_n \sin\left(\frac{n\theta}{2}\right)\right], \tag{31}$$

   and

$$\sigma_{\theta 3} = \mu \sum_{n=1}^{\infty} \frac{n}{2} r^{n/2-1}\left[-A_n \sin\left(\frac{n\theta}{2}\right) + B_n \cos\left(\frac{n\theta}{2}\right)\right]. \tag{32}$$

2. The stresses are of the form $\sigma_{(\cdot 3)} \sim r^{\frac{n}{2}-1}$. Hence near the crack tip, the term with $n = 1$ will dominate all the other terms since it is singular. The crack tip singularity is seen to be $O(r^{-1/2})$. The stress are predicted to be infinite in the elastic solution at the crack tip.

3. To complete the solution one needs to determine the A_n and B_n (for $n \geq 1$) using the boundary conditions at $r = R$.

4. If we consider the pure mode-III loading of a far field stress $\boldsymbol{\sigma} = \sigma_{23}^{\infty}(\boldsymbol{e}_2 \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_2)$, the displacement fields needs to be anti-symmetric and all $A_n = 0$. In this case, the crack tip stresses are of the form $\sigma_{ij} = K_{III} f_{ij}(\theta)/\sqrt{2\pi r} +$ bounded terms, where K_{III} depends solely on the geometry and the load. K_{III} is called the mode-III stress intensity factor. For a given geometry and load, it can be computed and then compared to the critical stress intensity factor $K_{III,c}$, which is a material property to determine if the crack will grow. For tensile loads, one also has K_I, the mode-I stress intensity factor, and for shear in the plane on has K_{II}, the mode-II stress intensity factor. For these latter two stress intensity factors one also has tabulated values of critical values.

5. It may seem non-physical that the stresses are infinite at the crack tip and thus one may get the impression that the solution is of little practical value. Notwithstanding, the elastic solution is quite useful. If one considers a small circle around the crack tip of radius $r = \epsilon$, then one can compute the energy available to flow to the crack tip should the crack try to move through the material in the 1-direction as

$$J = \boldsymbol{e}_1 \cdot \int_{-\pi}^{\pi} \left(W(\boldsymbol{\varepsilon})\mathbf{1} - \nabla \boldsymbol{u}^T \cdot \boldsymbol{\sigma}^T \right) \cdot \boldsymbol{n} \epsilon \, d\theta \,. \tag{33}$$

   This value turns out to be finite despite the stress singularity. If this energy is greater than the resistance provided by the material (the critical energy release rate), then the crack will advance. Equation (33) is the celebrated J-integral discovered by Jim Rice in the 1960s. It forms the basis of modern fracture mechanics.