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Deformation Gradient in Polar Coordinates: $\boldsymbol{F} = \boldsymbol{g}_i \otimes \boldsymbol{G}^i$

1 Point mapping

The deformation map $\boldsymbol{\varphi}$ maps points (R, Θ, Z) to points (r, θ, z), where

$$r(R, Z) = R + u(R, Z) \tag{1}$$

$$\theta(R, \Theta, Z) = \Theta + \phi(R, Z) \tag{2}$$

$$z(R, Z) = Z + w(R, Z)\,. \tag{3}$$

In the expressions above the coordinates of the points are radius, angle, and elevation – the standard polar coordinates.

2 Position vector

The position vector for the reference state is given by

$$\boldsymbol{X} = R\boldsymbol{E}_R(\Theta) + Z\boldsymbol{E}_Z\,. \tag{4}$$

The position vector for the deformed state is given by

$$\boldsymbol{x} = r\boldsymbol{e}_r(\theta) + z\boldsymbol{e}_z\,. \tag{5}$$

The basis vectors in these two expressions are the standard polar basis vectors. We employ upper and lower case symbols to remind ourselves that the upper case basis vectors are those at (R, Θ, Z) and the lower case basis vectors are those at (r, θ, z).

3 Tangent vectors

The tangent vectors to the coordinate lines are

$$\boldsymbol{G}_R = \frac{\partial \boldsymbol{X}}{\partial R} = \boldsymbol{E}_R \tag{6}$$

$$\boldsymbol{G}_\Theta = \frac{\partial \boldsymbol{X}}{\partial \Theta} = R\boldsymbol{E}_\Theta \tag{7}$$

$$\boldsymbol{G}_Z = \frac{\partial \boldsymbol{X}}{\partial Z} = \boldsymbol{E}_Z\,, \tag{8}$$

where we think of $\boldsymbol{X}$ being successively a parametric curve of R, Θ, and Z with the other two coordinates held fixed. The dual vectors are defined such that $\boldsymbol{G}^i \cdot \boldsymbol{G}_j = \delta^i_j$, which implies $\boldsymbol{G}^i = G^{ij}\boldsymbol{G}_j$ where the (inverse) metric components are $G^{ij} = [\boldsymbol{G}_i \cdot \boldsymbol{G}_j]^{-1}$:

$$\boldsymbol{G}^R = \boldsymbol{E}_R \tag{9}$$

$$\boldsymbol{G}^\Theta = \frac{1}{R}\boldsymbol{E}_\Theta \tag{10}$$

$$\boldsymbol{G}^Z = \boldsymbol{E}_Z\,. \tag{11}$$

The tangent vectors to the convected coordinate lines in the spatial configuration are

$$\boldsymbol{g}_r = \frac{\partial \boldsymbol{x}}{\partial R} = (1 + u_{,R})\boldsymbol{e}_r + r\phi_{,R}\boldsymbol{e}_\theta + w_{,R}\boldsymbol{e}_z \tag{12}$$

$$\boldsymbol{g}_\theta = \frac{\partial \boldsymbol{x}}{\partial \Theta} = r\boldsymbol{e}_\theta \tag{13}$$

$$\boldsymbol{g}_z = \frac{\partial \boldsymbol{x}}{\partial Z} = u_{,Z}\boldsymbol{e}_r + r\phi_{,Z}\boldsymbol{e}_\theta + (1 + w_{,Z})\boldsymbol{e}_z\,. \tag{14}$$

4 The deformation gradient

The deformation gradient maps tangent vectors from the reference to the deformed configuration. Given that $\boldsymbol{G}^i \cdot \boldsymbol{G}_j = \delta^i_j$, the requisite expression is

$$\boldsymbol{F} = \boldsymbol{g}_i \otimes \boldsymbol{G}^i\,. \tag{15}$$

Using the expressions for the spatial tangent vectors and the reference dual vectors yields in the mixed $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z\}$–$\{\boldsymbol{E}_R, \boldsymbol{E}_\Theta, \boldsymbol{E}_Z\}$ two-point basis:

$$[\boldsymbol{F}] = \begin{bmatrix} 1 + u_{,R} & 0 & u_{,Z} \\ r\phi_{,R} & 1 + \frac{u}{R} & r\phi_{,Z} \\ w_{,R} & 0 & 1 + w_{,Z} \end{bmatrix}. \tag{16}$$