Questions on Plasticity

1. **One dimensional return mapping algorithm** Assuming a material with both (linear) kinematic and isotropic hardening, use the following algorithm for calculating the material response to the following strain history

$$\varepsilon(t) = 0.01\sin(t) \qquad t \leq 30 \tag{1}$$

(1) Data Base $\{\varepsilon^p_n, \alpha_n, q_n\}$.

(2) Given Strain $\varepsilon_{n+1} = \varepsilon_n + \Delta\varepsilon_n$.

(3) Compute an "elastic trial state"

$$\sigma^{tr}_{n+1} = E(\varepsilon_{n+1} - \varepsilon^p_n) \tag{2}$$
$$q^{tr}_{n+1} = q_n \tag{3}$$
$$\alpha^{tr}_{n+1} = \alpha_n \tag{4}$$

(3) Test this trial state for yield

$$f^{tr}_n = |\sigma^{tr}_{n+1} - q^{tr}_{n+1}| - (\sigma_Y + H\alpha^{tr}_{n+1}) \tag{5}$$

If $f^{tr}_n \leq 0$ then step is elastic and $(\cdot)_{n+1} = (\cdot)^{tr}_n$ where $(\cdot)$ is one of $\{\sigma,\ q,\ \alpha\}$; else, the step is plastic. Let

$$\Delta\gamma = \frac{f^{tr}_n}{E + H + K}$$
$$\sigma_{n+1} = \sigma^{tr}_{n+1} - \Delta\gamma \text{sign}(\sigma^{tr}_{n+1} - q^{tr}_{n+1}) \tag{6}$$
$$\varepsilon^p_{n+1} = \varepsilon^p_n + \Delta\gamma \text{sign}(\sigma^{tr}_{n+1} - q^{tr}_{n+1})$$
$$q_{n+1} = q_n + \Delta\gamma K \text{sign}(\sigma^{tr}_{n+1} - q^{tr}_{n+1}) \tag{7}$$
$$\alpha_{n+1} = \alpha_n + \Delta\gamma$$

(a)Assume that $E = 30 \times 10^6$ psi, $\sigma_Y = 60 \times 10^3$ psi, $H = 40 \times 10^3$ psi, $K = 60 \times 10^3$ psi and choose a time increment to provide sufficient resolution. Make plots of σ vs. t, α vs. t, q vs. t, and σ vs. ε.

This algorithm is known as a return mapping algorithm. It is based on a backward Euler integration of the flow rules, the requirement that $f_{n+1} = 0$ during plastic flow, and the observation that $\text{sign}(\sigma_{n+1} - q_{n+1}) = \text{sign}(\sigma^{tr}_{n+1} - q^{tr}_{n+1})$.

(b)Prove this last "assertion" by subtracting (7) from (6) and making use of the facts that $\Delta\gamma > 0$, $E + K > 0$, and for any real number $x = |x|\text{sign}(x)$.

2. **Plastic Dissipation** Consider a closed stress cycle $\sigma(t)$ over a time period $[0, T]$ – i.e. one where $\sigma(0) = \sigma(T)$, which also implies that $\varepsilon^e(0) = \varepsilon^e(T)$. Show that the dissipation in the cycle which is defined as

$$\mathcal{D}_{\text{cycle}} := \int_0^T \sigma \dot{\varepsilon} \, dt \tag{8}$$

can be expressed as

$$\mathcal{D}_{\text{cycle}} = \int_0^T \sigma \dot{\varepsilon}^p \, dt \, . \tag{9}$$

This allows one to conclude that the dissipation rate in an elasto-plastic solid is given by $\sigma \dot{\varepsilon}^p$.

3. **One dimensional combined hardening** Consider a one-dimensional plastic response with kinematic and isotropic (linear) hardening. Find the stress rate response in terms of the strain rate.

4. **One dimensional kinematic and isotropic hardening** Consider the following model:

$$f(\sigma, q, \alpha) = |\sigma - q| - (\sigma_Y + H\alpha) \tag{10}$$
$$\dot{q} = K\dot{\gamma}\text{sign}(\sigma - q) \tag{11}$$
$$\dot{\varepsilon}^p = \dot{\gamma}\text{sign}(\sigma - q) \tag{12}$$
$$\dot{\alpha} = \dot{\gamma} \tag{13}$$

(a) Find $\dot{\gamma}$ during plastic flow

(b) Find $\dot{\sigma}$ for $\dot{\gamma} = 0$ and $\dot{\gamma} \neq 0$.

(c) Plot $\sigma(t)$, $\alpha(t)$, $q(t)$, and σ versus ε. Assume

$$E = 30 \times 10^6 \tag{14}$$
$$\sigma_Y = 60 \times 10^3 \tag{15}$$
$$H = 40 \times 10^3 \tag{16}$$
$$K = 60 \times 10^3 \tag{17}$$
$$\varepsilon(t) = 0.01 \sin(t) \tag{18}$$
$$0 \leq t \leq 30 \tag{19}$$

Use the following algorithm:

$$\sigma^{tr} = E(\varepsilon_{n+1} - \varepsilon^p_n)$$

If $f(\sigma^{tr}, q_n, \alpha_n) \leq 0$, then the time step is elastic and $\sigma_{n+1} = \sigma^{tr}$, $q_{n+1} = q_n$, $\alpha_{n+1} = \alpha_n$, and $\varepsilon^p_{n+1} = \varepsilon^p_n$. Else,

$$\Delta\gamma = \frac{f(\sigma^{tr}, q_n, \alpha_n)}{E + H + K} \tag{20}$$
$$q_{n+1} = q_n + \Delta\gamma K \text{sign}(\sigma^{tr} - q_n) \tag{21}$$
$$\varepsilon^p_{n+1} = \varepsilon^p_n + \Delta\gamma \text{sign}(\sigma^{tr} - q_n) \tag{22}$$
$$\alpha_{n+1} = \alpha_n + \Delta\gamma \tag{23}$$
$$\sigma_{n+1} = E(\varepsilon_{n+1} - \varepsilon^p_{n+1}) \tag{24}$$

5. **Nonlinear isotropic hardening** Modify the plasticity algorithm from Problem 4 to handle the case of zero kinematic hardening ($K = 0$) and **non-linear** isotropic hardening.

   Stress Strain:
   $$\sigma = E(\varepsilon - \varepsilon^p)$$
   Flow Rules:
   $$\dot{\varepsilon}^p = \dot{\gamma}\text{sign}(\sigma) \tag{25}$$
   $$\dot{\alpha} = \dot{\gamma} \tag{26}$$
   Yield Condition:
   $$f(\sigma, \alpha) = |\sigma| - (\sigma_y + H[1 - \exp(-\alpha)])$$
   Kuhn-Tucker Conditions:
   $$\dot{\gamma} \geq 0 \qquad f \leq 0 \qquad \dot{\gamma} f = 0$$
   Consistency:
   $$\dot{\gamma}\dot{f} = 0$$
   Note that your equation for the consistency parameter $\Delta\gamma$ will be non-linear, and you will need to use an iterative method to solve for it at each time step – Newton's method is a good choice. Assume $E = 30 \times 10^6$ psi, $\sigma_y = 60 \times 10^3$ psi, $H = 40 \times 10^3$ psi, and $\varepsilon(t) = 0.01 \sin(\beta t)$ for $0 \leq t \leq 30$ sec, where $\beta = 1$ rad/sec. Make three plots: σ versus t, α versus t, and σ versus ε.

6. **Nonlinear isotropic hardening** Modify the plasticity algorithm given in Problem 4 to work for zero kinematic hardening ($K = 0$) AND non-linear isotropic hardening.

   Stress Strain:
   $$\sigma = E(\varepsilon - \varepsilon^p)$$
   Flow Rules:
   $$\dot{\varepsilon}^p = \dot{\gamma}\text{sign}(\sigma) \tag{27}$$
   $$\dot{\alpha} = \dot{\gamma} \tag{28}$$
   Yield Condition:
   $$f(\sigma, \alpha) = |\sigma| - (\sigma_y + H[1 - \exp(-100\alpha)])$$

Kuhn-Tucker Conditions:

$$\dot{\gamma} \geq 0 \qquad f \leq 0 \qquad \dot{\gamma} f = 0$$

Consistency:

$$\dot{\gamma} \dot{f} = 0$$

Step (1) Develop an algorithm by applying a Backward Euler integration rule to the flow rules. Step (2) show the sign of the trial stress is the same as the sign of the actual stress. Step (3) enforce consistency at the end of the time step, if yielding is currently taking place; i.e. require

$$f(\sigma_{n+1}, \alpha_{n+1}) = 0$$

This provides a non-linear equation for $\Delta\gamma$; solve it by using a Newton iteration loop. Assume $E = 30 \times 10^6$ psi, $\sigma_y = 60 \times 10^3$ psi, $H = 40 \times 10^3$ psi, and $\varepsilon(t) = 0.01 \sin(\beta t)$ for $0 \leq t \leq 30$ sec, where $\beta = 1$ rad/sec. Make plots of σ and α versus time and σ versus ε.

7. **1D Nonlinear power-law hardening** Consider the following 1-D plasticity model with power-law hardening

$$\begin{aligned} \sigma &= E(\varepsilon - \varepsilon^p) && (29) \\ f(\sigma, \alpha) &= |\sigma| - (\sigma_Y + H\alpha^n) && (30) \\ \dot{\varepsilon}^p &= \dot{\gamma} \frac{\partial f}{\partial \sigma} && (31) \\ \dot{\alpha} &= \dot{\gamma}\,. && (32) \end{aligned}$$

Show that during active plastic loading that

$$\dot{\gamma} = \frac{E \text{sign}(\sigma) \dot{\varepsilon}}{E + Hn\alpha^{n-1}}\,. \qquad (33)$$

Use this result to find and expression for $\dot{\sigma}$ during active plastic loading in terms of $\dot{\varepsilon}, \alpha$ and the material parameters E, H and n.

8. **Duvaut-Lions Model** The one-dimensional Duvaut-Lions model for viscoplasticity is obtained from our rate-independent plasticity model by replacing the Kuhn-Tucker conditions and the plastic strain rate expression by the single (constitutive relation):

$$\dot{\varepsilon}^p = \frac{1}{\mu E}[\sigma - \bar{\sigma}]$$

where μ is a fluidity parameter and $\bar{\sigma}$ represents the solution to the rate-independent plasticity equations. By considering the relation

$$\dot{\sigma} = E[\dot{\varepsilon} - \dot{\varepsilon}^p]$$

develop an efficient algorithm for the integration of the Duvaut-Lions model that uses the solution to the rate-independent case as a known quantity. Check that your algorithm gives the correct limits: $\Delta t/\mu \to 0$ elastic response and $\Delta t/\mu \to \infty$ inviscid plastic response. [Hint: Exactly integrate exponential terms, approximate other terms.]

9. **Tresca versus Hencky-von Mises** The state of stress at a point is given as

$$\boldsymbol{\sigma} \rightarrow \begin{bmatrix} 70 & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & -35 \end{bmatrix} \frac{\text{N}}{\text{mm}^2}$$

   What are the algebraically largest and smallest values of σ_{22} for elastic behavior according to Tresca's yield condition? According to von Mises' yield condition? Assume the uniaxial yield stress is $\sigma_Y = 170\ \text{N/mm}^2$ and that both yield criteria are calibrated using a uniaxial test.

10. J_2 **Plasticity with combined hardening** Consider the case of J_2 plasticity. Compute $\frac{\partial f}{\partial \sigma_{ij}}$ for the case of linear kinematic plus linear isotropic hardening and determine $\dot{\gamma}$.

11. **Volume preservation of plastic flow** Consider a von Mises yield condition $f(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}'\| - \sigma_Y$ and associated flow. Show that the plastic strains are volume preserving – i.e. show that $\text{tr}[\boldsymbol{\varepsilon}^p] = 0$.

12. **Crack tip plastic zone** For elastic bodies with cracks the stress fields near the crack tips have singularities which indicates that yielding always takes place near the tip of a crack for any applied loads. For the case of a crack loaded in tension, as shown below, the stress field near the right-crack tip is given by

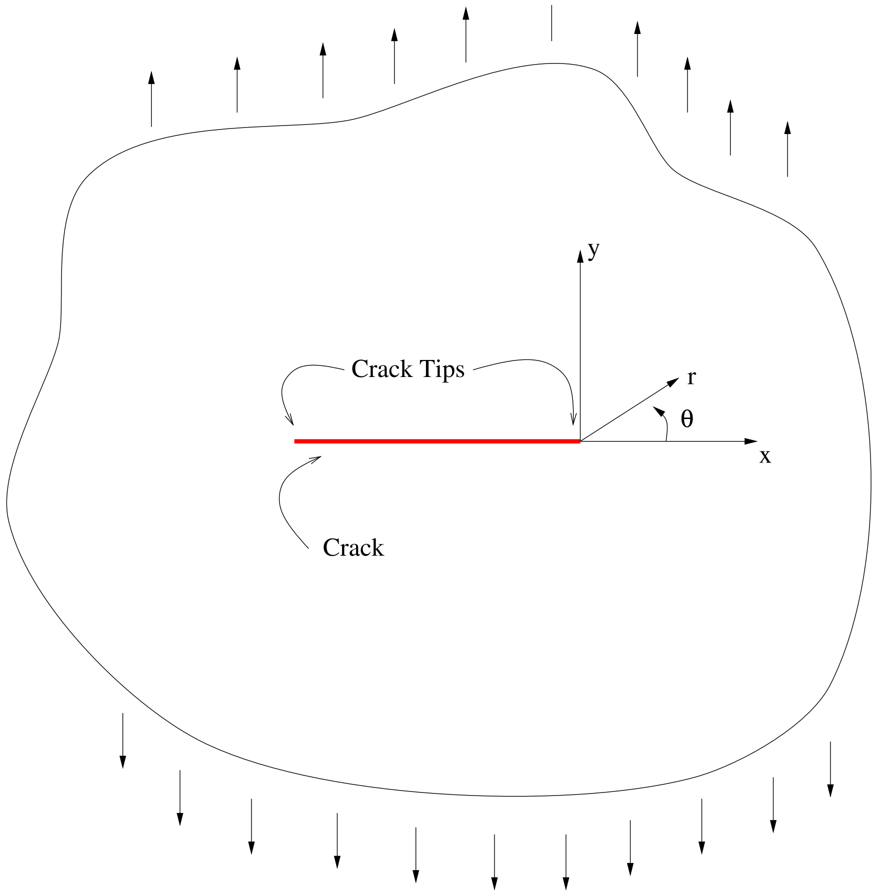
$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \tag{43}$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right] \tag{44}$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos\frac{3\theta}{2} \tag{45}$$

   where K_I is a geometry dependent parameter called the stress intensity factor. The coordinate origin is taken at the crack tip.

   Assume a material governed by von Mises' yield condition ($f(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}'\| - \sqrt{\frac{2}{3}}\sigma_Y$). Determine and plot the approximate shape of the plastic zone around the crack tip for plane strain and generalized plane stress. You may assume that at the interface between the elastic and plastic zones that the stress field given above is reasonably accurate. Also you may assume that Poisson's ratio $\nu = 0.3$. Normalize all distances in your plot by $3K_I^2/2\pi\sigma_Y^2$.



13. **Combined Kinematic and Isotropic Hardening** Consider the following plasticity model:

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij} \tag{46}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon^e_{kl} \tag{47}$$

$$f(\boldsymbol{\sigma}, \boldsymbol{q}, \alpha) = \|\boldsymbol{\sigma'} - \boldsymbol{q}\| - \sqrt{\frac{2}{3}}(\sigma_Y + H\alpha) \tag{48}$$

$$\dot{q}_{ij} = K\dot{\gamma}\frac{\partial f}{\sigma_{ij}} \tag{49}$$

$$\dot{\varepsilon}^p_{ij} = \dot{\gamma}\frac{\partial f}{\sigma_{ij}} \tag{50}$$

$$\dot{\alpha} = \dot{\gamma} \tag{51}$$

(a) Find $\frac{\partial f}{\sigma_{ij}}$

(b) Find $\dot{\gamma}$ during plastic flow.

14. **Non-associated flow** Consider the following plasticity model, where $\boldsymbol{q}$ is a vector of internal variables.

Stress-Strain:

$$\boldsymbol{\sigma} = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p)$$

Flow Rules (not necessarily associative):

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}^p &= \dot{\gamma}\boldsymbol{r}(\boldsymbol{\sigma}, \boldsymbol{q}) \\ \dot{\boldsymbol{q}} &= -\dot{\gamma}\boldsymbol{h}(\boldsymbol{\sigma}, \boldsymbol{q}) \end{aligned}$$

where $\boldsymbol{r}$ and $\boldsymbol{h}$ are given functions of stresses and internal variables $\boldsymbol{\sigma}$ and $\boldsymbol{q}$. Yield Function/Elastic domain:

$$\begin{aligned} &f(\boldsymbol{\sigma}, \boldsymbol{q}) \\ &\mathbb{E}_\sigma = \{(\boldsymbol{\sigma}, \boldsymbol{q}) \mid f(\boldsymbol{\sigma}, \boldsymbol{q}) \leq 0\} \end{aligned}$$

Kuhn-Tucker Conditions:

$$\dot{\gamma} \geq 0, \qquad f(\boldsymbol{\sigma}, \boldsymbol{q}) \leq 0, \qquad \dot{\gamma} f(\boldsymbol{\sigma}, \boldsymbol{q}) = 0$$

Consistency Condition:

$$\dot{\gamma}\dot{f}(\boldsymbol{\sigma}, \boldsymbol{q}) = 0$$

1. Find an expression for $\dot{\gamma}$ during plastic flow.
2. The elastoplastic modulus $\mathbb{C}^{ep}$ is defined through the relation $\dot{\boldsymbol{\sigma}} = \mathbb{C}^{ep} : \dot{\boldsymbol{\varepsilon}}$. Show that

$$\mathbb{C}^{ep} = \begin{cases} \mathbb{C} & \text{if } \dot{\gamma} = 0 \\ \mathbb{C} - \dfrac{\mathbb{C} : \boldsymbol{r} \otimes \mathbb{C} : \frac{\partial f}{\partial \boldsymbol{\sigma}}}{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbb{C} : \boldsymbol{r} + \frac{\partial f}{\partial \boldsymbol{q}} \cdot \boldsymbol{h}} & \text{if } \dot{\gamma} > 0 \end{cases}$$

15. **Plastic work based hardening** Consider the following alternative characterization of isotropic hardening. Let the yield function by given by

$$f(\boldsymbol{\sigma}, W^p) = \|\boldsymbol{\sigma}'\| - (\sigma_Y + \hat{\sigma}(W^p))$$

where W^p is the total plastic work; i.e.

$$W^p = \int_{-\infty}^{t} \boldsymbol{\sigma}(\beta) : \frac{d\boldsymbol{\varepsilon}^p}{d\beta}(\beta)\, d\beta$$

and $\hat{\sigma}(W^p)$ may in general be a non-linear function.

Assuming associated flow, $\dot{\boldsymbol{\varepsilon}}^p = \dot{\gamma}\partial f / \partial \boldsymbol{\sigma}$, find an expression for $\dot{\gamma}$ during plastic flow. Notes:

1. It is OK to have the term $d\hat{\sigma}/dW^p$ in your answer.
2. Recall Leibniz's rule:

$$\frac{d}{dt}\int_{f(t)}^{g(t)} h(t,s)\, ds = h(t, g(t))\frac{dg}{dt} - h(t, f(t))\frac{df}{dt} + \int_{f(t)}^{g(t)} \frac{\partial h}{\partial t}(t,s)\, ds$$

16. **Drucker-Prager** Consider the following yield function

$$f(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}' : \boldsymbol{\sigma}'\| - (\sigma_Y - k\,\mathrm{tr}[\boldsymbol{\sigma}])\,, \tag{61}$$

where k is a material constant. Assume an associative flow rule and find the plastic strain rate.

17. **Drucker-Prager** Consider a Drucker-Prager yield surface:

$$f(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}'\| - (\sigma_Y - Kp)$$

where σ_Y and K are material constants and $p = \frac{1}{3}\sigma_{ii}$ is the presssure.

1. Assuming an associated flow rule $\dot{\varepsilon}^p_{ij} = \dot{\gamma}\partial f/\partial\sigma_{ij}$, show that

$$\dot{\varepsilon}^p_{ij} = \dot{\gamma}\left(\frac{\sigma'_{ij}}{\|\boldsymbol{\sigma}'\|} + \frac{K}{3}\delta_{ij}\right)\,.$$

2. Assuming that the strain history $\varepsilon_{ij}(t)$ is known, find an expression for the consistency parameter, $\dot{\gamma}$, during plastic loading.
3. Assume a stress of the form

$$\boldsymbol{\sigma} \sim \sigma_Y t \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

and find the time t at which yield begins; assume that $K = \frac{1}{4}$.
4. Find the direction of the plastic strain rate at this moment.

18. **Drucker-Prager** Consider the Drucker-Prager yield condition

$$f(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}'\| - \sqrt{\frac{2}{3}}(\sigma_Y - \mu p)\,,$$

where $\boldsymbol{\sigma}' = [\mathbb{I} - (1/3)\mathbf{1}\otimes\mathbf{1}] : \boldsymbol{\sigma}$ is the stress deviator, $p = (\mathbf{1} : \boldsymbol{\sigma})/3$ is the pressure, and σ_Y and μ are material parameters.

1. Find the flow direction assuming associative plastic flow $\dot{\boldsymbol{\varepsilon}}^p = \gamma\partial f/\partial\boldsymbol{\sigma}$; i.e. compute $\partial f/\partial\boldsymbol{\sigma}$.
2. Assume a value of $\mu = 1/3$. If the stress is given by

$$\boldsymbol{\sigma} \sim \sigma_Y \cdot t \cdot \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & -2 \\ 3 & -2 & 3 \end{bmatrix}\,,$$

find the time t at which yield begins.

3. Determine the direction of plastic flow at this moment.

19. **Dissipation in Prandtl-Reuss model** Consider Prandtl-Reuss plasticity where the dissipation (rate) is given by $\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p$.

    1. Show that $\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{e}}^p = \boldsymbol{\sigma}' : \dot{\boldsymbol{e}}^p = \boldsymbol{\sigma}' : \dot{\boldsymbol{\varepsilon}}^p$
    2. By noting that Prandtl-Reuss applies to isotropic materials, show that during plastic flow $\dot{\gamma} = \boldsymbol{\sigma}' : \dot{\boldsymbol{e}}/\sqrt{2}k_M$. ($k_M$ is the calibration constant in von Mises condition.)

20. **Dissipation rate for associated von Mises flow** Show for perfect plasticity (i.e. no hardening) that the rate of dissipation can be expressed as

$$\mathcal{D} = \dot{\gamma}\sqrt{\frac{2}{3}}\sigma_Y = \dot{\bar{e}}^p \sigma_Y$$

    for the von Mises yield condition and associative flow.

21. **Kinematic hardening** Consider the following plasticity model:

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p \tag{85}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}^e \tag{86}$$

$$f(\boldsymbol{\sigma}, \boldsymbol{q}) = \|\boldsymbol{\sigma}' - \boldsymbol{q}\| - \sqrt{\frac{2}{3}}\sigma_Y \tag{87}$$

$$\dot{q}_{ij} = K\dot{\gamma}\frac{\partial f}{\partial \sigma_{ij}} \tag{88}$$

$$\dot{\varepsilon}_{ij}^p = \dot{\gamma}\frac{\partial f}{\partial \sigma_{ij}} \tag{89}$$

    where C_{ijkl}, K, and σ_Y are known constants. Suppose for a known constant β and time $t \geq 0$, that

$$\boldsymbol{\sigma}'(0) \to \frac{\sigma_Y}{3}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\boldsymbol{q}(0) \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

    and

$$\dot{\boldsymbol{\sigma}}'(t) \to \frac{\beta t}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

    (a) Show that $\boldsymbol{\sigma}'(t)$ remains on the yield surface $\forall t > 0$.

    (b) Find an expression for $\dot{\boldsymbol{q}}(t)$ in terms of known quantities and $\boldsymbol{q}$; explicitly write out the expression for $\dot{q}_{11}(t)$.

22. **Back stress Evolution** Consider the model in Problem 13. At time $t = 0$ the state of stress at a material point is

$$\boldsymbol{\sigma} \to \begin{bmatrix} 9000 & 0 & 0 \\ 0 & 5000 & 0 \\ 0 & 0 & 3100 \end{bmatrix} \text{psi} \tag{90}$$

and for $t \geq 0$ the state of stress is increasing at the constant rate

$$\dot{\boldsymbol{\sigma}} \to \begin{bmatrix} 20 & 0 & 0 \\ 0 & 45 & 0 \\ 0 & 0 & 40 \end{bmatrix} \text{psi/sec}\,. \tag{91}$$

The kinematic hardening constant is $K = 250$ (psi) and the isotropic hardening constant is $H = 0$. The yield stress $\sigma_Y = 40000\sqrt{3}$ (psi). Assume, $\boldsymbol{q} = \mathbf{0}$ at $t = 0$.

1. At what time t_1 does the material at this point begin to yield?
2. At time t_1, what is $\dot{\boldsymbol{q}}$.
3. Set up three simultaneous nonlinear ordinary differnetial equations for the non-zero components of $\boldsymbol{q}$ for time $t \geq t_1$. Plot their solution for 12 (sec) beyond initial yield. [Hint: (1) Use numerical integration. (2) Do not try and apply the return mapping algorithm.]

23. **Back stress evolution** Consider the model in Problem 13. At time $t = 0$ the state of stress at a material point is

$$\boldsymbol{\sigma} \to \begin{bmatrix} 38000 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{psi} \tag{92}$$

and for $t \geq 0$ the state of stress is increasing at the constant rate

$$\dot{\boldsymbol{\sigma}} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 0 & -1000 \end{bmatrix} \text{psi/sec}\,. \tag{93}$$

The kinematic hardening constant is $K = 250$ (psi) and the isotropic hardening constant is $H = 0$. The yield stress $\sigma_Y = 40000$ (psi). Assume, $\boldsymbol{q} = \mathbf{0}$ at $t = 0$.

1. At what time t_1 does the material at this point begin to yield?
2. At time t_1, what is $\dot{\boldsymbol{q}}$?
3. Compute the tragectory of the elastic domain in the π-plane. [Hint: (1) The center of the elastic domain is governed by a system of non-linear ODES. (2) Do not try to use the return mapping algorithm. The statement of this problem is stress driven so it is more easily approached from a direct manipulation of the governing equations. (3) Integrate the non-linear ODEs using the built-in time integrator in Matlab or Mathematica.]

24. **Yield surface evolution with kinematic hardening** An associative plasticity model with linear kinematic hardening and a von Mises yield condition. At time $t = 0$ assume the state of stress at a material point is

$$\boldsymbol{\sigma} \rightarrow \begin{bmatrix} \zeta\sigma_Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{94}$$

and for $t \geq 0$ the state of stress is increasing at the constant rate

$$\dot{\boldsymbol{\sigma}} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & -c \end{bmatrix}, \tag{95}$$

where $\zeta = 0.95$ and $c > 0$ is a given constant. Further assume, $\boldsymbol{q} = \boldsymbol{0}$ at $t = 0$.

Qualitatively, but with reasonable precision, describe the time evolution of the center of the elastic zone – i.e. the trajectory of $\boldsymbol{q}$ as a function of time in the π-plane (the octahedral plane). Make sure to identify all important moments in time.

25. **Upper Bound Theorem for Plastic Collapse** Consider a body Ω that is subjected to surface tractions $c\bar{t}_i$ on $\partial\Omega_t$, where $\bar{t}_i$ represents a load pattern and c is a scaling factor for the load. Assume zero body forces. A classical question is to estimate the value of c associated with gross plastic deformation in the body – the situation typically associated with collapse.

In this problem you will prove the upper bound theorem for plastic collapse. I will provide the steps and you need to provide the reasoning for each step. To start, first note that the Cauchy-Schwarz inequality tells us that $|\boldsymbol{A} : \boldsymbol{B}| \leq \|\boldsymbol{A}\| \cdot \|\boldsymbol{B}\|$ for any two second order tensors.

We begin by assuming a (virtual) displacement field $\delta\boldsymbol{w}$ such that $\delta\boldsymbol{w}$ is zero on $\partial\Omega_u$. Then it follow by the Cauchy-Schwarz inequality that

$$\boldsymbol{\sigma}' : \boldsymbol{\varepsilon}(\delta\boldsymbol{w}) \leq |\boldsymbol{\sigma}' : \boldsymbol{\varepsilon}(\delta\boldsymbol{w})| \leq \|\boldsymbol{\sigma}'\| \cdot \|\boldsymbol{\varepsilon}(\delta\boldsymbol{w})\|, \tag{96}$$

where the notation $\boldsymbol{\varepsilon}(\boldsymbol{v}) \equiv (1/2)[\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T]$ defines the symmetric gradient – i.e. the strain field associated with a generic displacement field $\boldsymbol{v}$.

Since $\delta\boldsymbol{w}$ is virtual we can arbitrarily assume that its symmetric gradient is traceless (i.e. deviatoric), resulting in

$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta\boldsymbol{w}) \leq \|\boldsymbol{\sigma}'\| \cdot \|\boldsymbol{\varepsilon}(\delta\boldsymbol{w})\|. \tag{97}$$

(a) Show how (97) follows from (96).

Assume now that the material obeys they von Mises yield condition. This implies

$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta\boldsymbol{w}) \leq \sqrt{\frac{2}{3}}\sigma_Y \|\boldsymbol{\varepsilon}(\delta\boldsymbol{w})\|. \tag{98}$$

(b) Show how (98) follows from (97).

We can now integrate (98) over the body to give

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta \boldsymbol{w}) \, dV \leq \int_\Omega \sqrt{\frac{2}{3}} \sigma_Y \|\boldsymbol{\varepsilon}(\delta \boldsymbol{w})\| \, dV \tag{99}$$

It can then be shown that

$$c \int_{\partial\Omega_t} \bar{\boldsymbol{t}} \cdot \delta \boldsymbol{w} \, dA \leq \int_\Omega \sqrt{\frac{2}{3}} \sigma_Y \|\boldsymbol{\varepsilon}(\delta \boldsymbol{w})\| \, dV \, . \tag{100}$$

(c) Show how (100) follows from (99). [Hint: weak equilibrium]

Equation (100), implies by simple division that the load factor can be no greater than a specific value:

$$c \leq \frac{\int_\Omega \sqrt{\frac{2}{3}} \sigma_Y \|\boldsymbol{\varepsilon}(\delta \boldsymbol{w})\| \, dV}{\int_{\partial\Omega_t} \bar{\boldsymbol{t}} \cdot \delta \boldsymbol{w} \, dA}$$

The utility of this result is very wide ranging. Pick *any* virtual displacement field and the right hand side provides and upper bound to the load factor. By judiciously picking virtual motions that are near the true collapse motion of system, one can obtain quite good estimates for the collapse loads.

26. **Plastic Torsion** In the Prandtl stress formulation of torsion one has that

$$\sigma_{13} = \Psi_{,2} \qquad \text{and} \qquad \sigma_{23} = -\Psi_{,1} \, .$$

   1. Show that if the material yields according to the von Mises condition, then

$$\|\nabla \Psi\| - \sqrt{\frac{1}{3}} \sigma_Y \leq 0 \, .$$

   [The $\sqrt{1/3}$ is NOT a typo.]

   2. At the limit load, every point on the cross-section will have yielded and thus $\|\nabla\Psi\| = \sqrt{1/3}\sigma_Y$ everywhere on the cross-section. This implies that the surface $\Psi(x_1, x_2)$ has the property that its gradient has constant norm everywhere.

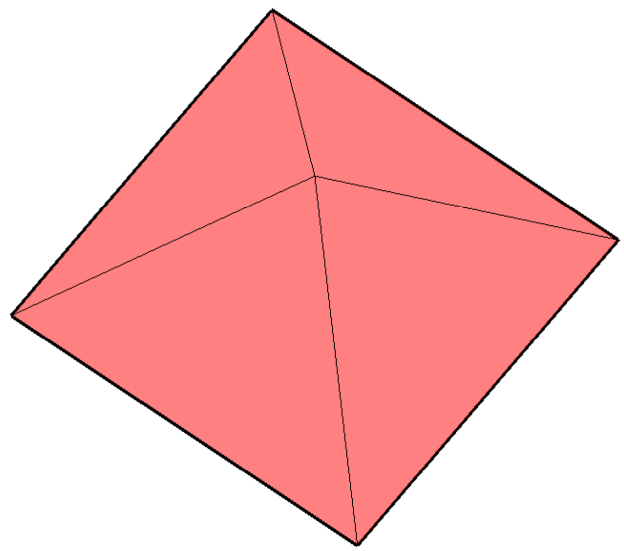
      (a) For a torsion rod with circular cross-section, this implies that Ψ is in the shape of a right-circular cone, where the height of the apex of the cone is $\sqrt{1/3}\sigma_Y R$, where R is the radius of the rod. Using this information compute the ultimate torque (limit load) using the relation

$$T = \int_A x_1 \sigma_{23} - x_2 \sigma_{13} \, dA = \int_A -x_1 \Psi_{,1} - x_2 \Psi_{,2} \, dA$$

      OR the relation

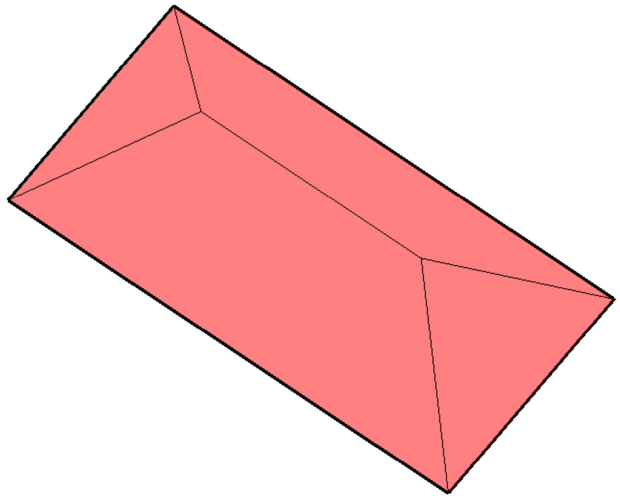
$$T = 2 \int_A \Psi \, dA.$$

(b) Compute the ultimate torque (limit load) for rod with square cross-section $a \times a$ using one of the two previous relations. Note that Ψ will be in the shape of flat facets that must be pieced together to form a continuous surface, where on each facet the norm of the gradient of Ψ has the same constant value $\sqrt{1/3}\sigma_Y$. The figure below depicts this situation. [Hint: You can perform the integral over only a part of the cross-section and then multiply by an appropriate factor to account for the entire cross-section.]

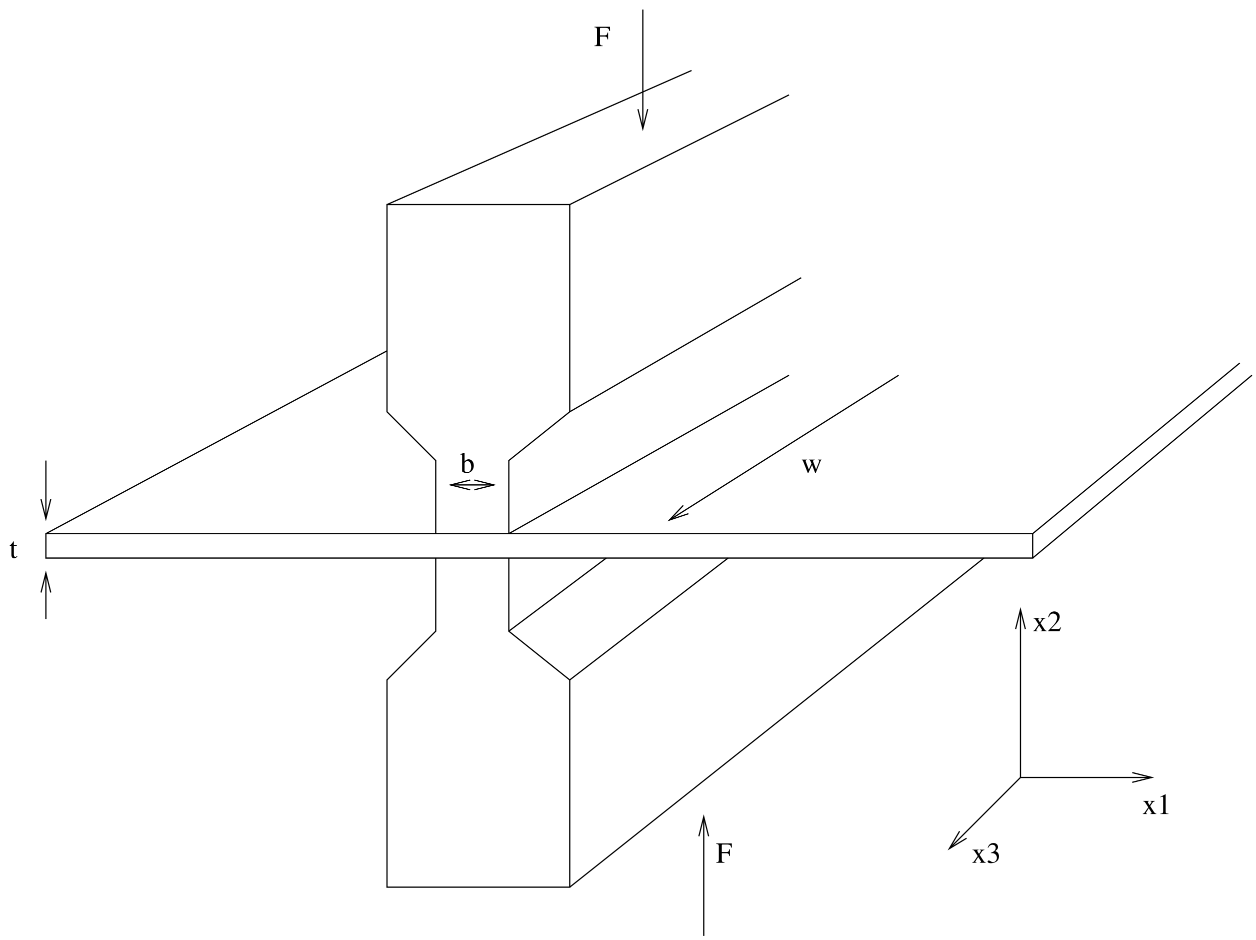


A common way of thinking about this surface is to imagine what shape a "heap" of sand would take if you poured a bunch of sand on top of a board cut out in the shape of the cross-section. This is known as the *sand-pile* or *sand-heap* analogy.

3. (Extra) Compute the ultimate torque for a rectangular bar with cross-sectional dimensions $a \times b$, where $b > a$. The function Ψ looks like:



27. **Material testing** The plane strain compression test is one of the better methods for obtaining a compressive stress-strain curve for a metallic material. A schematic figure is show below.



The indenting dies have a width w and a breadth b. The specimen has a width w and a thickness t. Recommended ratios for (w/b) and (b/t) for such a test are $(w/b) > 6$ and $2 < (b/t) < 4$. As a load is applied the metal between the indenters is prevented from moving in the $\boldsymbol{e}_3$ direction by the constraint from the unstressed material adjacent to the indented region. This gives a state of (approximate) plane strain. **Neglecting** the effects of **elasticity** and friction, show that during plastic deformation in this mode:

(a) the flow strength Y of the material is related to the mean pressure $p = F/bw$ under the dies by

$$Y = \frac{\sqrt{3}}{2} p$$

and

(b) that the equivalent plastic strain

$$\bar{e}^p = \frac{2}{\sqrt{3}} \ln\left(\frac{t_i}{t_f}\right)$$

where $\dot{\bar{e}}^p = \sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}^p : \dot{\boldsymbol{\varepsilon}}^p}$, and t_i and t_f are the initial and final thicknesses (under the dies) of the strip specimen. [Assume J_2 plasticity for parts (a) and (b).]

28. **Plastically loaded thick-walled cylinder** Consider a thick walled cylinder subject to an internal pressure p_i and an external pressure p_o; internal radius r_i and external radius r_o; plane strain conditions $\frac{\partial}{\partial z}(\cdot) = 0$ and $u_z = 0$; made of an elastic perfectly plastic Tresca material with Tresca constant k_T.

(1) Determine the expressions for all the stresses when the cylinder is entirely elastic.

(2) Find the condition on the applied pressures for initiation of yield.

(3) Determine the radial and hoop stresses when the elastic-plastic interface is located at $r = R$.

(4) Determine the relation for computing the elastic-plastic interface radius when the applied pressures are known.

[Note that the compatibility equation here is the same in the sphere problem; i.e. $\varepsilon_{rr} = (r\varepsilon_{\theta\theta})_{,r}$.

For part 5 and 6, assume that $p_o = 0$ and $\frac{r_o}{r_i} = 2$.

(5) Plot σ_{rr}/k_T, $\sigma_{\theta\theta}/k_T$, and σ_{zz}/k_T vs. r/r_i when $p_i = {}^1\!/_2 k_T$.

(6) Plot σ_{rr}/k_T and $\sigma_{\theta\theta}/k_T$ vs. r/r_i when $R/r_i = 3/2$.