

Questions on Viscoelasticity

- For a strain history of $\varepsilon(t) = \varepsilon_0 \sin(\omega t)$ the steady state response of the Standard Linear Solid may be written as $\sigma(t) = \varepsilon_0 [E'(\omega) \sin(\omega t) + E''(\omega) \cos(\omega t)]$. E' is called the storage modulus and E'' is called the loss modulus. The loss tangent is defined to be $\tan(\delta) = E''/E'$. The moduli E' and E'' are functions of the frequency ω and hence so is the loss tangent. Assume $E_0 = 11$, $E_\infty = 1$, and $\eta = 1$. Plot E' , E'' , and $\tan(\delta)$ versus $\log(\omega)$ over the range $\omega \in (10^{-2}, 10^3)$.
- For a strain history of $\varepsilon(t) = \varepsilon_0 \sin(\omega t)$ the response of the Generalized Standard Linear Solid may be written as $\sigma(t) = \sigma_0 [E'(\omega) \sin(\omega t) + E''(\omega) \cos(\omega t)]$. E' is known as the storage modulus and E'' is known as the loss modulus.
 - Assuming a GSLS with constants $E_\infty = E_1 = E_2 = 1$, $\eta_1 = 1$ and $\eta_2 = 100$ determine and plot $E'(\omega)$ versus $\log(\omega)$ over the range $\log(\omega) \in [-3, 3]$. Comment on the shape of the graph and interpret the distinctive features.
 - Let $\varepsilon(t) = t \sin(t)H(10 - t) + H(t - 10) - H(t - 20)$. Over the range $0 \leq t \leq 25$, plot $\varepsilon(t)$, $\sigma(t)$ and cross-plot σ vs. ε ; use the same properties as in part a. You may use an approximate method for determining the stresses.
- Consider a SLS with $E_\infty = 1$ MPa, $E_1 = 1$ MPa, and $\tau = 1$ sec. The material is subjected to the following strain history:

$$\varepsilon(t) = \begin{cases} 0 & t \leq 0 \\ \exp(t/A) & 0 < t \leq 1 \\ -(t/A)^2 + t/A & 1 < t \leq 2 \\ 0 & 2 < t, \end{cases}$$

where $A = 2$ sec. Use the Herrmann-Peterson recursion relation to compute and plot the stress response in the time interval $T = [0, 5]$ sec. Make a plot of stress versus time as well as a cross-plot of stress versus strain.

- GSLS** Consider a 1-D GSLS that is subject to a strain history of $\varepsilon(t) = \varepsilon_0 \sin(\omega t)$. The steady state stress response can be written as $\sigma(t) = \varepsilon_0 [E'(\omega) \sin(\omega t) + E''(\omega) \cos(\omega t)]$, where $E'(\omega)$ is the storage modulus and $E''(\omega)$ is the loss modulus. Find the expressions for the storage and loss moduli for a GSLS with two Maxwell elements. Assume $E_\infty = E_1 = E_2 = 1$, $\eta_1 = 1$, and $\eta_2 = 100$; thus the relaxation modulus/function/kernel is $E(t) = 1 + 1 \exp[-t] + 1 \exp[-t/100]$. Plot $E'(\omega)$, $E''(\omega)$, and $\tan(\delta)$ versus $\log_{10}(\omega)$ over the range $[-3, 3]$. Interpret the distinctive features of the curves.
- Consider the Standard Linear Solid in 1-D. Assume that the strain history is known to be $\varepsilon(t) = H(t) \sin(\omega t)$ and find the stress as a function of time. Do this in two different ways: (a) directly solve the governing ODE and (b) apply a Laplace transform method to the ODE; use a table to do the inverse Laplace transform.

6. **Differential Equation Visco Elastic Model** Consider the following viscoelastic material model:

$$p_0\varepsilon + p_1\frac{d\varepsilon}{dt} + p_2\frac{d^2\varepsilon}{dt^2} + p_3\frac{d^3\varepsilon}{dt^3} = q_0\sigma + q_2\frac{d^2\sigma}{dt^2} + q_4\frac{d^4\sigma}{dt^4},$$

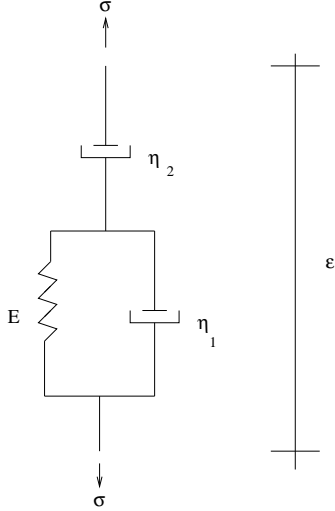
where q_j and p_k are given constants. Determine expressions for the storage and loss moduli.

7. You are given the following viscoelastic constitutive relation:

$$A\frac{d^3\varepsilon}{dt^3} + B\frac{d\varepsilon}{dt} + C\varepsilon = D\sigma + E\frac{d^2\sigma}{dt^2}.$$

Find expressions for the storage and loss moduli assuming that A, B, C, D and E are given constants.

8. Develop a 1-D version of the recursion relation method for a 1-D SLS. Assuming $E_\infty = 1$, $E_o = 2$, $\eta = 1$, and $\omega = 1$, plot the stress as a function of time for the strain history given in Problem 5 for $t \in [0, 12]$. Determine the maximum Δt that you can use to get reasonably accurate solutions by comparing to the exact solution from Problem 5.
9. Consider a Kelvin Solid in series with an elastic spring. What is the strain response of the material to a step loading of stress; i.e., find $\varepsilon(t)$ for a given $\sigma(t) = \sigma_o H(t)$. For the spring assume a spring constant of E_1 and for the Kelvin Solid assume the spring constant is E_2 and the dashpot viscosity is η_2 .
10. **Kelvin Solid** A Kelvin solid is subjected to an exponentially varying strain $\varepsilon(t) = \varepsilon_o[1 - e^{-t\gamma}]H(t)$, where ε_o and γ are given constants.
1. Find and plot the stress response $\sigma(t)$.
 2. Compute the work of the load up to an arbitrary time T .
11. **Kelvin Solid Creep Kernel** Find the creep kernel for the Kelvin solid.
12. The rheological model shown below has been proposed as a model for asphalt concrete. What is the governing stress-strain equation for this model? What do the storage and loss modulus curves look like for such a material?



13. Consider a Standard Linear Solid with material parameters E_∞ , E_o , and η . What is the creep function for this material? i.e. what is the strain response to an input step in the stress.
14. Consider a 1-D bar composed of a Kelvin Solid. The bar, $\Omega = (0, L)$, is built-in at $x = 0$ and is free at $x = L$. The bar is subjected to a body force field $b(x, t) = [H(t) - H(t - 1) + H(t - 2) - H(t - 3)]x$. Use weak equilibrium, an appropriate time discretization method, and a solution space $\widehat{\mathcal{S}}_n = \{u(x, t_n) \mid A_n x + B_n < x - L/4 > + C_n < x - L/2 > + D_n < x - 3L/4 >\}$ to develop an approximate method of solution for the displacement field of the bar. In particular, you should explicitly give expressions for the matrix equations governing (A_n, B_n, C_n, D_n) at any given time $t = t_n$. Be sure to fully define both the right and left hand sides. Ignore inertial terms. [Warning, the similar example from lecture has a number of algebra and calculus errors so be careful.]
Extra: Implement your scheme over the time interval $(0, 15)$ assuming $L = 1$, $E = 100$, and $\eta = 100$. Provide sufficient graphs to explain the behavior of the bar. Attach your code.
15. Consider a 1-D system with zero displacement boundary conditions at $x = 0$ and an imposed stress at $x = L$ of the form

$$\bar{\sigma} = \sigma_o[H(t - t_1) - H(t - t_2)]. \quad (36)$$

Find the elongation, $u(L, t)$, if the system is made of a Maxwell fluid.

16. Consider a 1-D bar occupying the region $[0, L]$. The bar is fixed at $x = 0$ and $x = L$ (i.e. $u(0, t) = u(L, t) = 0$). It is subjected to a body force $b(x, t) = C_1 x H(t)$. The bar is composed of a Maxwell fluid.

Assume: $L = 1$, $E = 1$, $\eta = 1$, $C_1 = 1$.

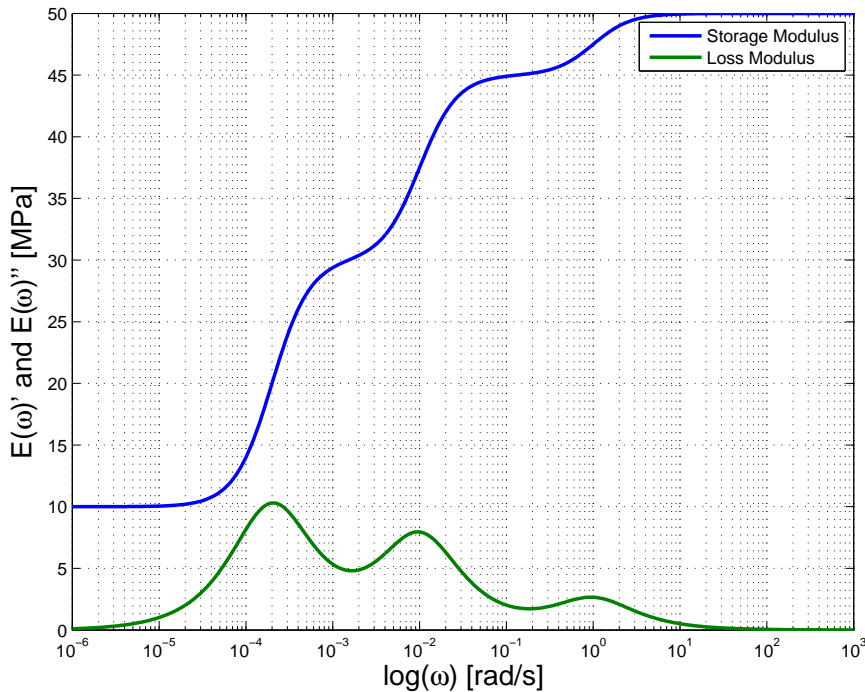
Find and plot $u(x, t)$ at times $t \in \{0, \Delta t, 2\Delta t, \dots, 6\Delta t\}$, where $\Delta t = 0.1\eta/E$. Solve the spatial part of the problem using an approximate weak form method with $\widehat{\mathcal{S}} =$

$\{u(x, t) \mid u(x, t_n) = A_n x(x - L)\}$. Solve the temporal part using a backward Euler method. Plot your results against the exact space-time answer.

17. Find and plot $u(x, t)$ at times $t \in \{0, \Delta t, 2\Delta t, \dots, 6\Delta t\}$, where $\Delta t = 0.1\eta/E$. Solve the spatial part of the problem using an approximate weak form method with $\hat{\mathcal{S}} = \{u(x, t) \mid u(x, t_n) = A_n x(x - L)\}$ and $\hat{\mathcal{V}} = \{\delta u(x) \mid \delta u(x) = \delta A x(x - L)\}$ Solve the temporal part using a backward Euler method. Plot your results against the exact space-time answer.

18. **GLS Estimation** Shown below is the graph of the storage and loss modulus for a viscoelastic material. You would like to model the material using a generalized standard linear solid model.

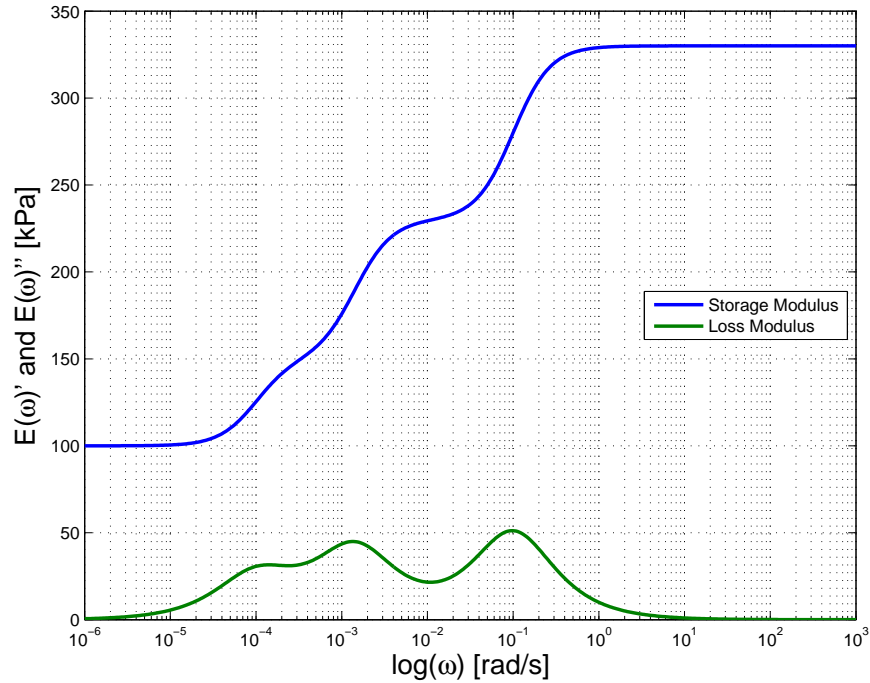
1. How many Maxwell elements will you need?
2. What are the associated relaxation times?
3. What is the long time modulus?
4. What are the moduli associated with the Maxwell elements?
5. What are the viscosities associated with the Maxwell elements?



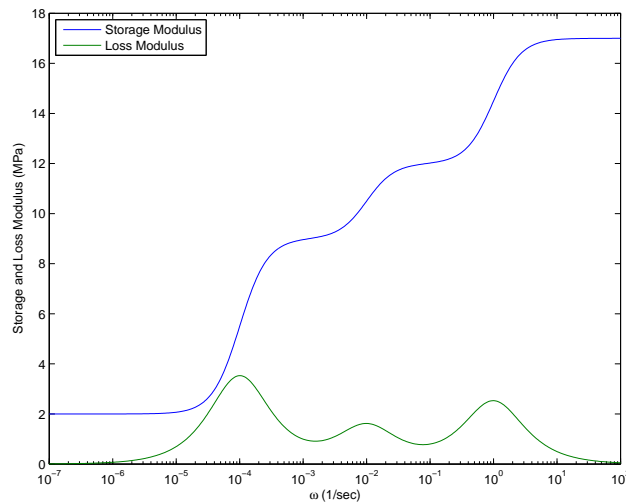
19. **GLS Estimation** Shown below is the graph of the storage and loss modulus for a viscoelastic material. You would like to model the material using a generalized standard linear solid model.

1. How many Maxwell elements will you need?

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4. What are the moduli associated with the Maxwell elements?
5. What are the viscosities associated with the Maxwell elements?



20. Consider a one-dimensional viscoelastic material whose storage and loss modulus are as given in the figure below. If the material is to be modeled using a Generalized Standard Linear Solid, how many relaxation elements will you need? and what will the constants be in the GSLS? Hint: It is sufficient to only consider the E' curve to answer this question.



21. In the accompanying file you will find stress data for a stress relaxation experiment. The stress response is generated by an imposed strain history

$$\varepsilon(t) = \begin{cases} 10^{-1}t & t < 10^{-1} \text{ s} \\ 10^{-2} & t \geq 10^{-1} \text{ s} . \end{cases}$$

This strain history involves a rise time of 100 ms to reach the hold strain of 10 mstrain. The first column of the data file gives a time value (in seconds) and the second column a corresponding stress value (in MPa). Note that the time spacing of the data points is not uniform.

Find an expression for the relaxation modulus $E(t)$. Make sure you use a suitable number of terms to adequately model the data. Note that the data file includes the rise time response.

22. **Relaxation Function: Proney Series Fitting** In the accompanying file you will find stress data for a stress relaxation experiment. The stress response is generated by an imposed strain history

$$\varepsilon(t) = \begin{cases} 10^{-1}t & t < 10^{-1} \text{ s} \\ 10^{-2} & t \geq 10^{-1} \text{ s} . \end{cases}$$

This strain history involves a rise time of 100 ms to reach the hold strain of 10 mstrain. The first column of the data file gives a time value (in seconds) and the second column a corresponding stress value (in MPa). Note that the time spacing of the data points is not uniform.

Find an expression for the relaxation modulus $E(t)$. Make sure you use a suitable number of terms to adequately model the data. Note that the data file includes the rise time response.

23. **Relaxation Function: Prony Series Fitting** In the accompanying file you will find stress data for a stress relaxation experiment (beware that the data include measurement noise). The stress response is generated by an imposed strain history

$$\varepsilon(t) = \begin{cases} 10^{-1}t & t < 10^{-1} \text{ s} \\ 10^{-2} & t \geq 10^{-1} \text{ s} . \end{cases}$$

This strain history involves a rise time of 100 ms to reach the hold strain of 10 mstrain. The first column of the data file gives a time value (in seconds) and the second column a corresponding stress value (in kPa). Note that the time spacing of the data points is not uniform.

Find an expression for the relaxation modulus $E(t)$. Make sure you use a suitable number of terms to adequately model the data. Note that the data file includes the rise time response; I suggest removing it but that is not necessary, if you do not want to do so.

24. In the accompanying file you will find stress data for a stress relaxation experiment in shear. The shear stress response is generated by an imposed engineering strain history

$$\gamma(t) = \begin{cases} 10^{-1}t & t < 10^{-2} \text{ s} \\ 10^{-3} & t \geq 10^{-2} \text{ s}. \end{cases}$$

This strain history involves a rise time of 10 ms to reach the hold strain of 1000 μ strain. The first column of the data file gives a time value and the second a corresponding shear stress value. The time spacing of the data points is not uniform.

1. Find an expression for the (shear) relaxation modulus $\mu(t)$. Make sure you use a suitable number of terms to adequately model the data.
2. Assume the overall material response is given by

$$\begin{aligned} \boldsymbol{\sigma} &= p\mathbf{1} + \mathbf{s} \\ \mathbf{s} &= \int_{-\infty}^{\hat{t}} 2\mu(\hat{t} - t)\dot{\boldsymbol{\epsilon}} dt \\ p &= K\theta, \end{aligned}$$

where the bulk modulus $K = 500$ MPa. Find the steady state stress response to a strain input of the form

$$\boldsymbol{\epsilon} \sim \sin(\omega t) \begin{bmatrix} 1 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \mu\text{strain}.$$

Plot the in-phase and out-of-phase parts of $\|\boldsymbol{\sigma}\|$ versus $\log(\omega)$ for $\omega \in [10^{-2}, 10^3]$. (When plotting make sure you use a sufficient number of points per logarithmic decade. In Matlab the command `logspace` is helpful in this regard.)

3. Consider now a round bar of length 200 mm and diameter 40 mm which is made of this material. The bar is subjected to an oscillating torsional load that induces a strain field in the bar of the form

$$\boldsymbol{\epsilon}(x, y, z, t) \sim \sin(\omega t) \begin{bmatrix} 0 & 0 & -y\beta \\ 0 & 0 & x\beta \\ -y\beta & x\beta & 0 \end{bmatrix},$$

where $\omega = 0.2$ [rad/sec], $\beta = \pi/2400$ [rad/mm], and z is the coordinate measure along the axis of the bar. How much energy is dissipated in the bar per cycle of oscillation. Assuming the heat capacity of the material is 2211 kJ/m³K, how much does the bar heat up per cycle? Assume the dissipated energy has time to uniformly redistribute in the bar and that there are no energy losses across the surfaces of the bar (i.e. assume adiabatic boundary conditions).

25. In elementary elastic beam theory we know that the governing equations are given by:

$$\begin{aligned} \theta &= w' & \kappa &= \theta' \\ M &= EI\kappa \\ \frac{dM}{dx} &= -V & \frac{dV}{dx} &= -q. \end{aligned}$$

Consider now a general viscoelastic material $\sigma(t) = \int_{-\infty}^t E(t-s) \frac{d\varepsilon}{ds} ds$.

1. Only one of the beam equations changes. Which one?
2. Using the fact that in the elementary theory, $\varepsilon = -y\kappa$, find this changed expression. Note, y is the depth coordinate on the cross-section of the beam.

26. **Viscoelastic Stress-Strain with WLF model** In the accompanying file you will find stress data for a stress relaxation experiment. The stress response is generated by an imposed strain history

$$\varepsilon(t) = \begin{cases} 10^{-1}t & t < 10^{-1} \text{ s} \\ 10^{-2} & t \geq 10^{-1} \text{ s}. \end{cases}$$

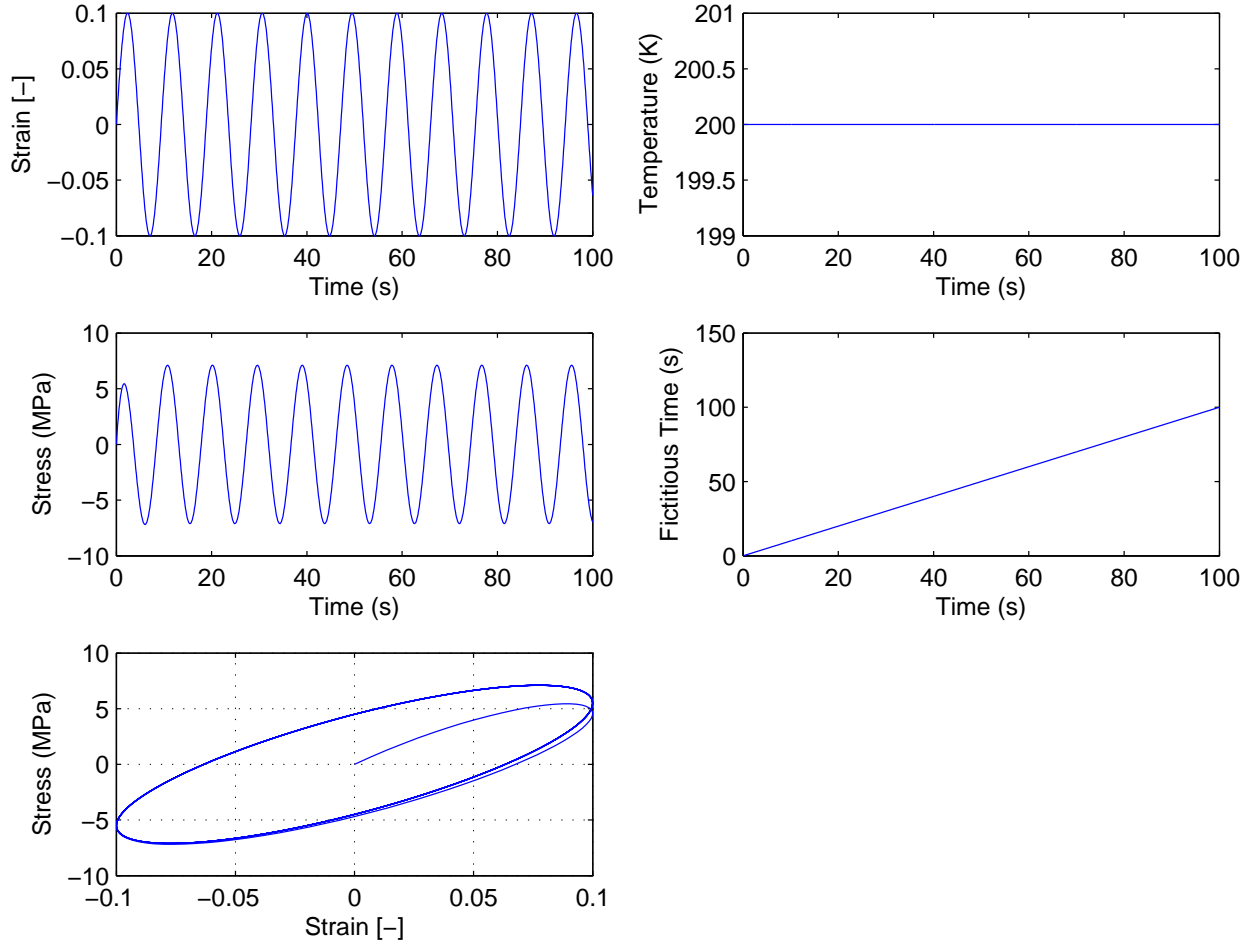
This strain history involves a rise time of 100 ms to reach the hold strain of 10 mstrain. The first column of the data file gives a time value (in seconds) and the second column a corresponding stress value (in MPa). Note that the time spacing of the data points is not uniform.

(a) Find an expression for the relaxation modulus $E(t)$. Make sure you use a suitable number of terms to adequately model the data. Note that the data file includes the rise time response.

(b) Consider a thermorheologically simple viscoelastic material with reference transition temperature $T_o = 200$ K. Further assume a WLF model for the shift factor

$$\log(a) = \frac{-C_1(T - T_o)}{C_2 + T - T_o},$$

with $C_1 = 20$ and $C_2 = 100$ K. Consider now that the material is subjected (from quiescent conditions) to a strain history of $\varepsilon(t) = 0.1 \sin(t/\tau_o)$ and a thermal history of $T(t) = T_o + \beta t + \gamma \cos(t/2\tau_o)$, where $\beta = 0.01$ K/s and $\gamma = 5$ K. Assume that the material is to be modeled as a standard linear solid with $E_\infty = 10$ MPa, $E_o = 100$ MPa, and $\tau_o = 1.5$ s and find the stress response over the time interval $I = [0, 100]$ s. Make the following plots (a) time versus strain, (b) time versus temperature, (c) time versus stress, (d) time versus fictitious time, and (e) stress versus strain. To help you debug your solution below is the answer to the question for the case where $\beta = \gamma = 0$



27. Show that 2-D plane strain viscoelastic problems can also be treated using a time dependent stress function. In particular assume the material law given below and find the governing relation for the stress function.

$$\mathbf{s} = \int_0^t 2\mu(t - \tau) \frac{d\mathbf{e}}{d\tau} d\tau \quad (58)$$

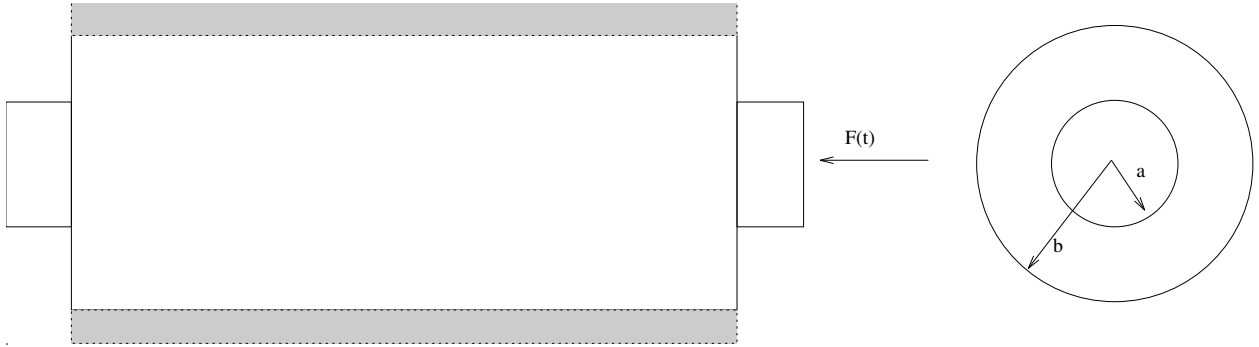
$$p = \int_0^t K(t - \tau) \frac{d\theta}{d\tau} d\tau, \quad (59)$$

where \mathbf{s} and \mathbf{e} are the stress and strain deviators, p is the pressure, and $\theta = \text{tr}[\boldsymbol{\varepsilon}]$ is the dilation. You may wish to consider the use of Laplace transforms.

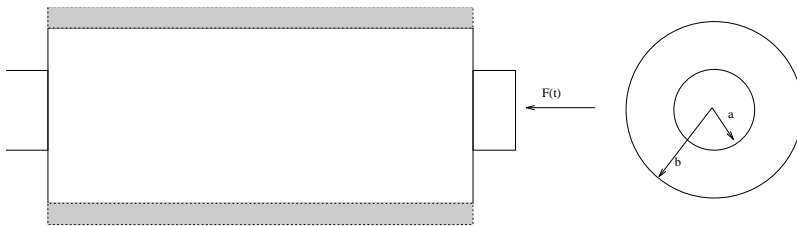
28. Shown is a long hollow cylinder of length L , inner radius a , and outer radius b . The cylinder is made of an isotropic material that is a Standard Linear Solid in its deviatoric behavior and elastic in its bulk behavior. The outer radius of this cylinder is fixed. The inner radius is bonded to a rigid rod of radius a . An arbitrary time varying force $F(t)$ is applied to the rod.

(1) Determine expressions for the stress and strain states in the cylinder as a function of time in terms of the applied force.

(2) Determine the expression for the displacement of the rigid rod as a function of time. [Neglect end-effects in your solution.]



29. Shown is a long hollow cylinder of length L , inner radius a , and outer radius b . The cylinder is made of an isotropic material that is Maxwell in its deviatoric behavior and elastic in its bulk behavior. The outer radius of this cylinder is fixed. The inner radius is bonded to a rigid rod of radius a . A force $F(t) = KH(t)$ is applied to the rod, where K is a given constant and $H(\cdot)$ is the Heaviside function. Determine the stress and strain states in the cylinder as a function of time. Also determine the expression for the displacement of the rigid rod. Neglect the end-effect in your solution. [Hints: (1) What does the solution look like if this was an elastic problem? (2) Try and determine σ_{rz} first.]



30. In viscoelastic materials the Poisson function is defined to be $\nu(t) = -\varepsilon_{22}(t)/\varepsilon_{11}(t)$ in a uniaxial constant load experiment – i.e. a cylinder of material is exposed to a step stress in the 1-direction and $\nu(t)$ is the ratio of the axial strain to the transverse strain. Assume a material that has elastic bulk behavior and Maxwell deviatoric behavior:

$$p = K\varepsilon_{ll} \quad (60)$$

$$\dot{\varepsilon}_{ij} = \frac{\dot{s}_{ij}}{2\mu} + \frac{s_{ij}}{\eta} \quad (61)$$

and find the Poisson function. Evaluate your answer for the two limits $\nu(0)$ and $\nu(\infty)$.

31. Consider material whose volumetric behavior is elastic $p = K\theta$ and whose deviatoric behavior is governed by a Kelvin Solid model. Assume a uniaxial state of stress (in the 1-direction) that is suddenly imposed at time $t = 0$ and determine the Poisson function $\nu(t) = -\varepsilon_{22}/\varepsilon_{11}$. Make a plot to illustrate the basic behavior.

32. Consider a material that responds elastically for volumetric deformations and as a SLS for deviatoric motions. A body composed of this material is subjected to a motion described by a displacement field of the following form:

$$\mathbf{u}(\mathbf{x}) = (\boldsymbol{\omega} \times \mathbf{x})H(t),$$

where $H(t)$ is the Heaviside step function and $\boldsymbol{\omega}$ is a constant given vector, such that $\|\boldsymbol{\omega}\| \ll 1$. Determine the stress response of the body.

33. Consider a linear isotropic viscoelastic rod with a non-circular cross-section. Assume linear viscoelasticity in convolution form for the deviatoric behavior and elastic bulk response; i.e.

$$\begin{aligned} \mathbf{s}(t) &= \int_{-\infty}^t 2\mu(t - \beta) \frac{d\mathbf{e}}{d\beta} d\beta, \\ p &= K \text{tr}[\boldsymbol{\varepsilon}]. \end{aligned}$$

Extend St. Venant's torsion theory to this case by assuming

$$\begin{aligned} u_x &= \alpha(t)\Phi(y, z) \\ u_y &= -\alpha(t)zx \\ u_z &= \alpha(t)yx. \end{aligned}$$

In particular

1. Show that there are only two non-zero stresses and that they are given by

$$\begin{aligned} \sigma_{xy} &= \left[\int_{-\infty}^t 2\mu(t - \beta) \frac{d\alpha}{d\beta} d\beta \right] \frac{1}{2} (\Phi_{,y} - z) \\ \sigma_{xz} &= \left[\int_{-\infty}^t 2\mu(t - \beta) \frac{d\alpha}{d\beta} d\beta \right] \frac{1}{2} (\Phi_{,z} + y) \end{aligned}$$

2. Show that Φ must satisfy Laplace's equation.
3. Show that

$$T(t) = \bar{J} \int_{-\infty}^t \mu(t - \beta) \frac{d\alpha}{d\beta} d\beta,$$

where you will need to give an appropriate definition to \bar{J} .

34. Consider St. Venant's theory of torsion for prismatic bars with arbitrary cross-sections but now for a material with a viscoelastic response $\sigma_{ij} = \int_{-\infty}^t 2\mu(t - s) d\varepsilon_{ij}/ds ds$ for the shears $i \neq j$.

1. Show that the warping function $\psi(x_1, x_2)$ still satisfies Laplace's equation $\nabla^2 \psi = 0$.

2. Show that the torque $T(t) = \frac{J_{\text{eff}}}{2} \int_{-\infty}^t 2\mu(t-s)d\alpha/ds ds$, where the effective polar moment of inertia has the same expression as in the elastic case.
35. **St. Venant viscoelastic torsion:** Consider a square bar with cross-section $a \times a$ and length L . The bar is made of a material whose volumetric response is elastic, $p = K\theta$, and whose deviatoric response is viscoelastic (standard linear solid). The bar is fixed at one end and the other end is subjected to an imposed step rotation, $\theta(t) = \theta_o H(t)$, which in turn imposes a step twist-rate $\alpha(t) = \alpha_o H(t)$ on the entire bar, where $\alpha_o = \theta_o/L$. Find an expression for the required torque on the end of the bar. Provide your answer in terms of α_o , the dimensions of the bar, and the parameters of the standard linear solid.