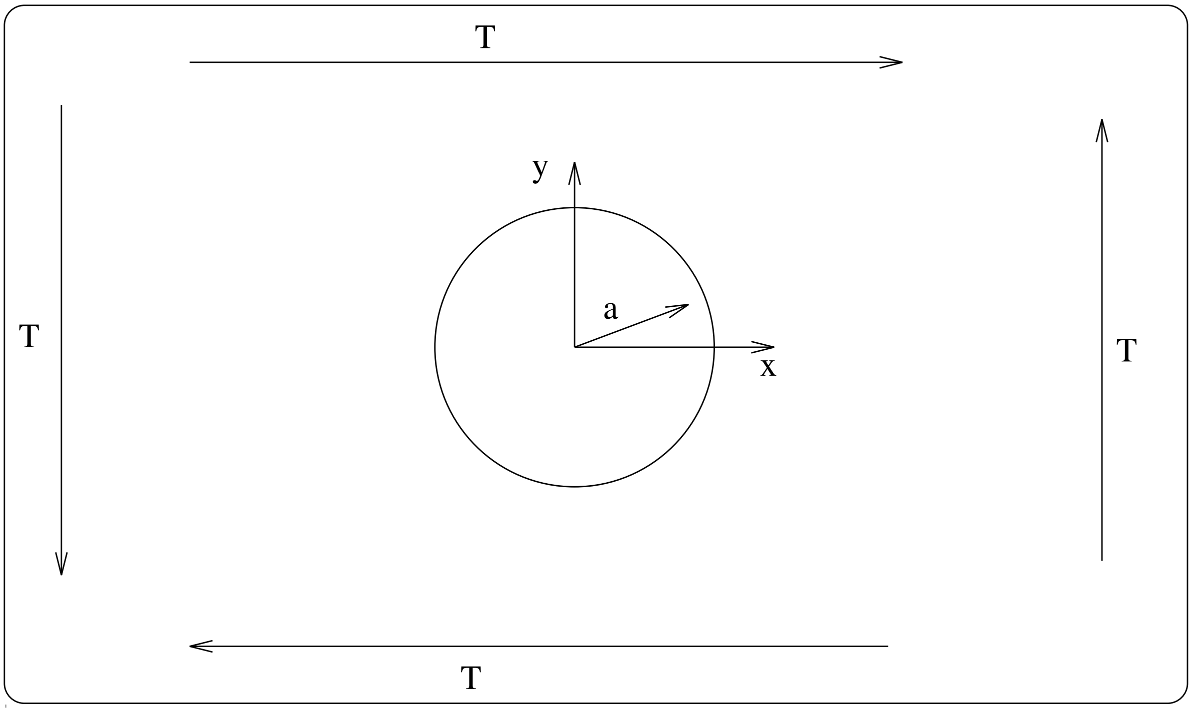
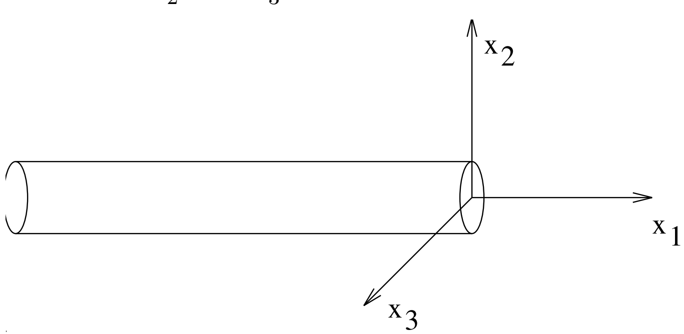
**Questions on Classical Solutions**

1. Consider an infinite linear elastic plate with a hole as shown. Uniform shear stress $\sigma_{xy} = T$ is applied at infinity. Determine the value of the stress $\sigma_{\theta\theta}$ on the edge of the hole. What is the stress concentration factor for this case?




[Hint: Superposition and appropriate terms from Mitchell.]

2. Consider a rod model with axial loads, shear loads in the $x_2$ and $x_3$ directions, bending about the $x_2$ and $x_3$ axes and torsion about the $x_1$ axis.




**Assume** a displacement field of the following form

$$u_1(x_i) = \bar{u}_1(x_1) - x_2\theta_3(x_1) + x_3\theta_2(x_1) \tag{1}$$
$$u_2(x_i) = \bar{u}_2(x_1) - x_3\theta_1(x_1) \tag{2}$$
$$u_3(x_i) = \bar{u}_3(x_1) + x_2\theta_1(x_1) \tag{3}$$

where $\bar{u}_i(x_1)$ is the displacement of the rod centroid at a location $x_1$ and $\theta_i(x_1)$ is the rotation of the cross-section at a location $x_1$ about the $i^{\text{th}}$ axis.

**Assume** the following stresses are zero: $\sigma_{22} = \sigma_{33} = \sigma_{23} = 0$. [Note, these assumptions are in contradiction to parts of the kinematic assumption.]

**Assume** the rod is prismatic and traction free on the lateral surfaces. Do **not** assume the body forces are zero.

**(1)** Determine the "Strain-Displacement" relations in terms of the kinematic fields $\bar{u}_i$ and $\theta_i$. Note, $\varepsilon_{22}$ and $\varepsilon_{33}$ will be in contradiction with the stress assumption.

**(2)** Define the force resultant on a cross-section as

$$\boldsymbol{N} = \int_A \boldsymbol{\sigma} \boldsymbol{e}_1 \, dA$$

Show that

$$\boldsymbol{N}_{,1} + \boldsymbol{q} = \boldsymbol{0}$$

by manipulating the definition of the force resultant and suitably defining $\boldsymbol{q}$ in terms of the body forces. In this part and part (3), the following application of the divergence theorem is useful

$$\int_A \sigma_{i2,2} + \sigma_{i3,3} \, dA = \int_{\partial A} \sigma_{i2} n_2 + \sigma_{i3} n_3 \, d\gamma$$

where $(0, n_2, n_3)^T$ is the normal to the perimeter of the cross-section.

**(3)** Define the moment resultant on a cross-section as

$$\boldsymbol{M} = \int_A (x_2 \boldsymbol{e}_2 + x_3 \boldsymbol{e}_3) \times \boldsymbol{\sigma} \boldsymbol{e}_1 \, dA$$

Show that

$$\boldsymbol{M}_{,1} + \boldsymbol{e}_3 \times \boldsymbol{N} + \boldsymbol{m} = \boldsymbol{0}$$

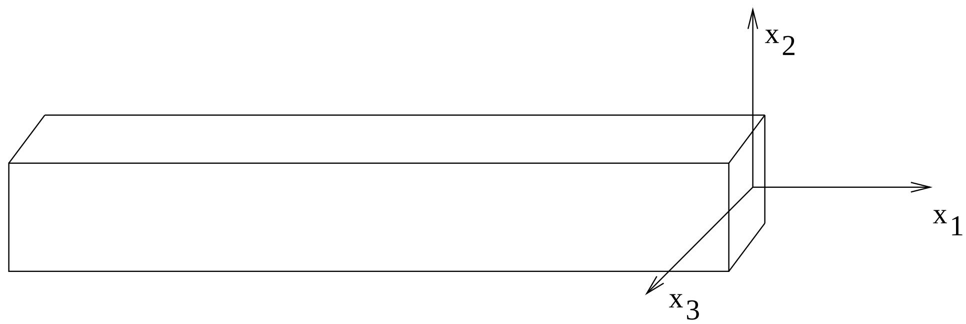
by manipulating the 3-D equilibrium equations and suitably defining $\boldsymbol{m}$ in terms of the body forces.

**(4)** Show that the integrated constitutive relations are given by

$$N_1 = EA\bar{u}_{1,1} \tag{4}$$
$$N_2 = GA\gamma_2 \tag{5}$$
$$N_3 = GA\gamma_3 \tag{6}$$
$$M_1 = GJ_1\theta_{1,1} \tag{7}$$
$$M_2 = EI_2\theta_{2,1} \tag{8}$$
$$M_3 = EI_3\theta_{3,1} \tag{9}$$

where $\gamma_2 = \bar{u}_{2,1} - \theta_3$, $\gamma_3 = \bar{u}_{3,1} + \theta_2$ (i.e. shear strains), $E$ is Young's modulus, $G$ is the shear modulus, $A$ is the cross-sectional area, $J_1$ is the polar moment of inertia about the $x_1$ axis, and $I_2$ and $I_3$ and the moments of inertia around the $x_2$ and $x_3$ axes, respectively.

3. Consider a linear elastic isotropic beam subjected to axial and shear loads in the $x_1$ and $x_2$ directions, and bending about the $x_3$ axis.




Develop a shear deformable beam theory for this system by **assuming** a displacement field of the following form

$$u_1(x_i) = \bar{u}_1(x_1) - x_2\theta(x_1) \tag{10}$$
$$u_2(x_i) = \bar{u}_2(x_1) \tag{11}$$
$$u_3(x_i) = 0\,, \tag{12}$$

where $\bar{u}_i(x_1)$ is the displacement of the beam centroid at a location $x_1$ and $\theta(x_1)$ is the rotation about the 3 axis of the cross-section at a location $x_1$.

**Assume** the following stresses are zero: $\sigma_{22} = \sigma_{33} = \sigma_{23} = \sigma_{13} = 0$. [Note, these assumptions are in contradiction to parts of the kinematic assumption.]

**Assume** the beam is prismatic and traction free on the lateral surfaces. Do **not** assume the body forces are zero.

**(1)** Determine the "Strain-Displacement" relations in terms of the kinematic fields $\bar{u}_i$ and $\theta$. Note, some of your strains will be in contradiction with the stress assumption.

**(2)** Define the axial force resultant on a cross-section as

$$P = \int_A \sigma_{11}\, dA \tag{13}$$

and the shear force resultant on the cross-section as

$$V = \int_A \sigma_{12}\, dA\,. \tag{14}$$

By integrating the equilibrium equations for the stresses over the cross-section, show that

$$P_{,1} + p = 0 \tag{15}$$
$$V_{,1} + q = 0\,. \tag{16}$$

Provide suitable definitions for the $p$ and $q$ and argue why they are appropriate.

**(3)** Define the moment resultant on the cross-section as

$$M = -\int_A x_2\sigma_{11}\, dA\,. \tag{17}$$

Consider the first moment of the equilibrium equations for the stresses and show

$$M_{,1} + V + m = 0 \tag{18}$$

Provide a suitable definition for $m$ and argue why it is appropriate. [The first moment of any quantity $f$, in this context, is simply $\int_A x_2 f$.]

**(4)** Show that the integrated constitutive relations are given by

$$P = EA\epsilon \tag{19}$$
$$V = GA\gamma \tag{20}$$
$$M = EI\kappa \tag{21}$$

where $\epsilon = \bar{u}_{1,1}$ (axial strain), $\gamma = \bar{u}_{2,1} - \theta$, (shear strain), $\kappa = \theta_{,1}$ (curvature), $E$ is the Young's modulus, $G$ is the shear modulus, $A$ is the cross-sectional area, and $I$ is the moment of inertia around the $x_3$ axis.

4. The stress function

$$\phi = A\left(\frac{xy}{4} - \frac{xy^2}{4h} - \frac{xy^3}{4h^2} + \frac{Ly^2}{4h} + \frac{Ly^3}{4h^2}\right)$$

is proposed as giving the solution for a cantilever beam ( $-h < y < h$, $0 < x < L$) loaded by a uniform shear along the top surface, the lower surface and the end $x = L$ being free from load. In what respects is this solution imperfect?

5. Consider the classical cantilever beam problem with a load, $P$, at its tip. Precisely formulate the boundary conditions for this problem within the context of 3D elasticity. Assume that the beam has a solid square cross-section of dimensions $2a \times 2a$ and a length $L$.

6. Consider a two dimensional plane-strain body occupying the region $\mathcal{B} = \{(X_1, X_2) \mid 0 \le X_1 \le a \text{ and } 0 \le X_2 \le b\}$. You are told that the stress field is described by the Airy stress function $\Phi = cr^3\sin(\theta)$, where $c$ is a known constant. What tractions are being applied to the body?

7. Consider the compatibility equation $\nabla \times \nabla \times \boldsymbol{\varepsilon} = 0$. Under the assumption that $u_1$ and $u_2$ are functions only of $x_1$ and $x_2$ and $u_3 = 0$ show that there is only one non-trivial compatibility relation $\varepsilon_{11,22} + \varepsilon_{22,11} - 2\varepsilon_{12,12} = 0$.

8. Consider the case of polar coordinates and the strains in polar coordinates. Assume that the displacement field is purely radial and independent of $\theta$ and $z$; i.e. $u_r = \hat{u}_r(r)$ and $u_\theta = u_z = 0$. Show that a necessary condition for compatibility is $\varepsilon_{\theta\theta,r} = \frac{1}{r}(\varepsilon_{rr} - \varepsilon_{\theta\theta})$.

9. Consider the anti-plane strain assumption of

$$u_1 = 0 \tag{22}$$
$$u_2 = 0 \tag{23}$$
$$u_3 = \hat{u}_3(x_1, x_2) \tag{24}$$

   Show that $\nabla^2 u_3 = 0$.

10. Consider a square linear elastic bar with cross-section 3 x 3 inches and length 24 inches. The shear modulus of the material is $12 \times 10^6$ psi.

    1. Assuming St. Venant torsion (i.e. free warping and appropriately applied end-tractions), at what location(s) on the cross-section does yield first start.
    2. Compute the torsional stiffness of the bar per unit length.

11. Consider an isotropic elastic square bar in torsion with unit length sides.

    (a) Compute $\bar{J}$ using the double trignometric solution and single trignometric solution; comment on the speed of convergence for the two series.

    (b) Assume a torque of $12 \times 10^3$ and a shear modulus of $12 \times 10^6$. Compute the twist rate, $\alpha$, for the bar and the maximum shear stress and its location.

    (c) Make a plot of the warped section.

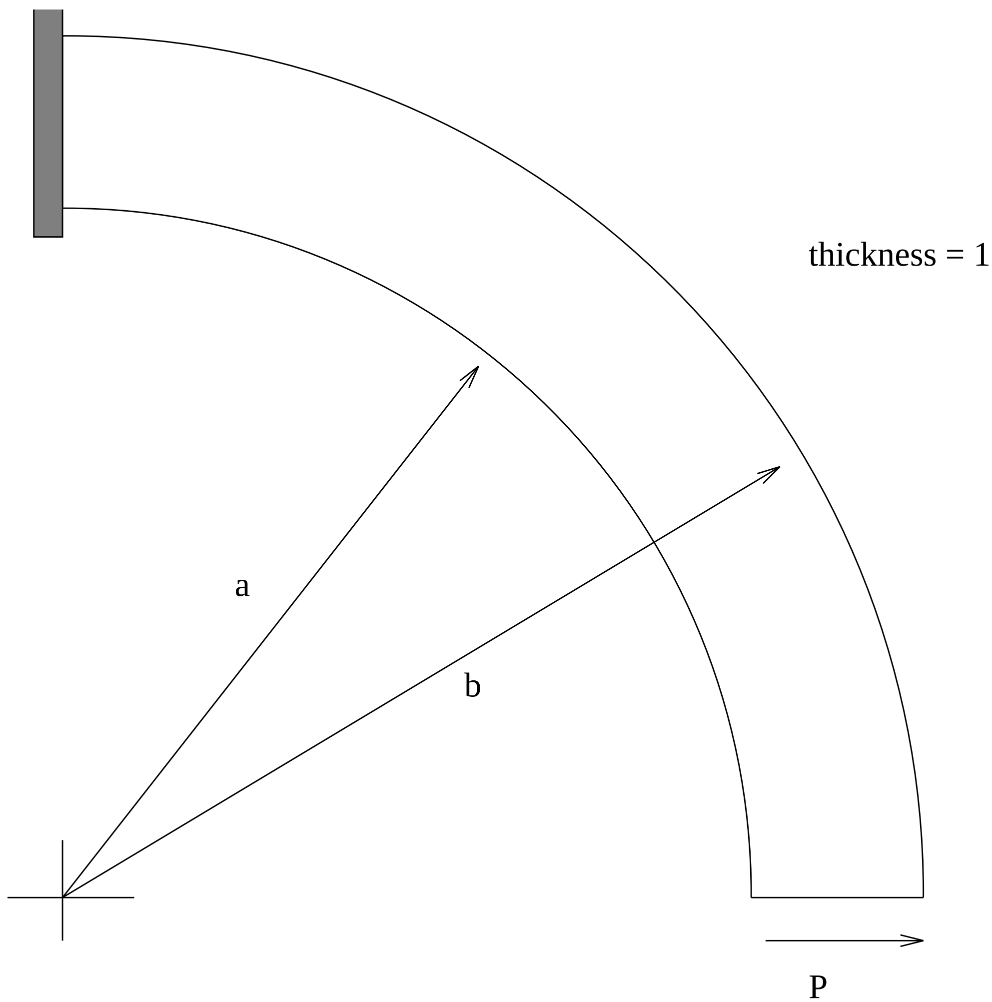
12. Consider an elliptical shaft in torsion with minor and major radii $a$ and $b$. Assume Prandtl's stress function has the form $\psi = C(y^2/a^2 + z^2/b^2 - 1)$.

    1. Find an expression for $C$.
    2. What is the torsional stiffness of the shaft per unit length.
    3. Express the maximum shear stress in terms of the applied torque and the geometry.

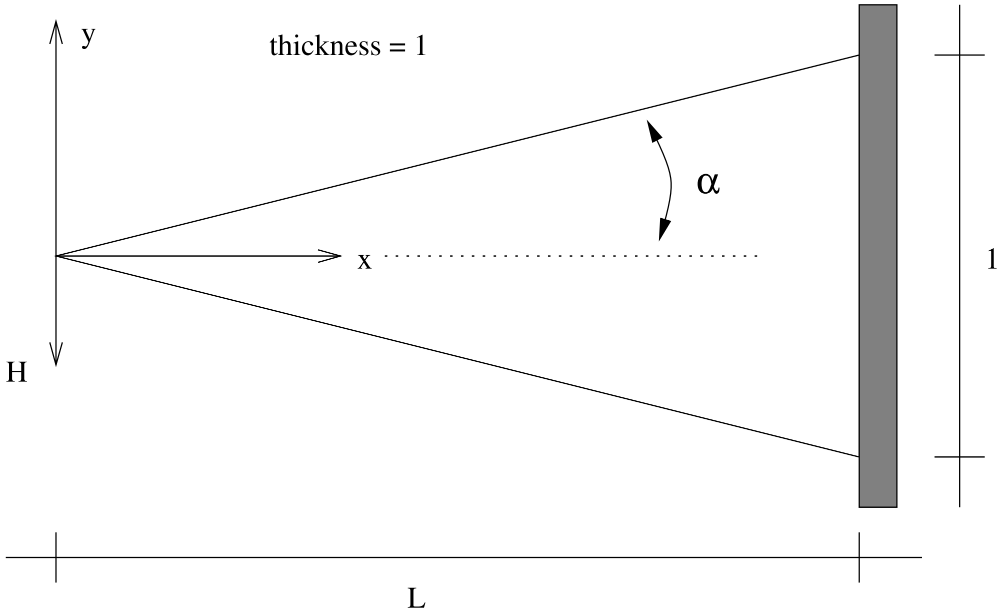
13. For the curved beam shown below: (a) Determine the stress field by starting with the stress function

$$\Phi = (Ar\ln(r) + Br^3 + C/r)\sin(\theta)$$

    (b) Set-up the linear equations that are used to find $A$, $B$, and $C$ in terms of $a$, $b$, and $P$. (c) At $\theta = \pi/2$ plot the normal (bending) stress on the section and compare it to the Bernoulli-Euler solution. Make plots for $(a, b) = (0.5, 2)$ and $(a, b) = (8.5, 10)$. Assume $P = 1$ for the plots. (d) Comment on the physical implication of $\sigma_{rr}$ in the curved beam. Relate it to the situation in a straight beam.




14. Consider the tapered beam shown below. (a) Determine the expression for $A$ in the stress function $\phi = Ar\theta\cos\theta$ in terms of $H$ and the angle $\alpha$. (b) Plot the bending and shear stresses at $x = L/2$ for $L = 1$ and $L = 10$. On the same figures, plot the Bernoulli-Euler parabolic solution for shear in a cantilever and the bending stresses in a cantilever; (see Popov or any other strength of materials text). Assume $H = 1$ for the plots.




15. A circular plane strain disk made of a linear elastic isotropic material is subjected to a radial pressure, $p(\theta) = \hat{\sigma}\cos(2\theta)$, on its perimeter.

    (a) Find the maximum hoop stress $\sigma_{\theta\theta}$.

    (b) Determine the hoop strain $\varepsilon_{\theta\theta}$.

16. Consider a 2-D (plane strain) isotropic linear elastic solid disk of radius 2. It is subjected to applied tractions at $r = 2$

$$\boldsymbol{t} = 2\boldsymbol{e}_r + 24\sin(2\theta)\boldsymbol{e}_\theta\,. \tag{66}$$

    Determine the state of stress and strain in the disk.

17. When we looked at the use of the Airy stress function to solve 2-D linear elastic isotropic plane strain problems we only examined the zero body force case. Suppose that we have a problem with body forces that emmenate from a known potential $V(x, y)$ such that $\boldsymbol{b} = -\nabla V$. Show that if we assume a stress function, $\phi$ such that

$$\begin{aligned}\sigma_{xx} &= V + \frac{\partial^2\phi}{\partial y^2}\\ \sigma_{yy} &= V + \frac{\partial^2\phi}{\partial x^2}\\ \sigma_{xy} &= -\frac{\partial^2\phi}{\partial x\partial y}\,,\end{aligned}$$

    then equilibrium is automatically satisfied. Further show that the governing equation for the stress function will be of the form

$$\nabla^4\phi = C\nabla^2 V\,.$$

    (You should determine what $C$ is.)

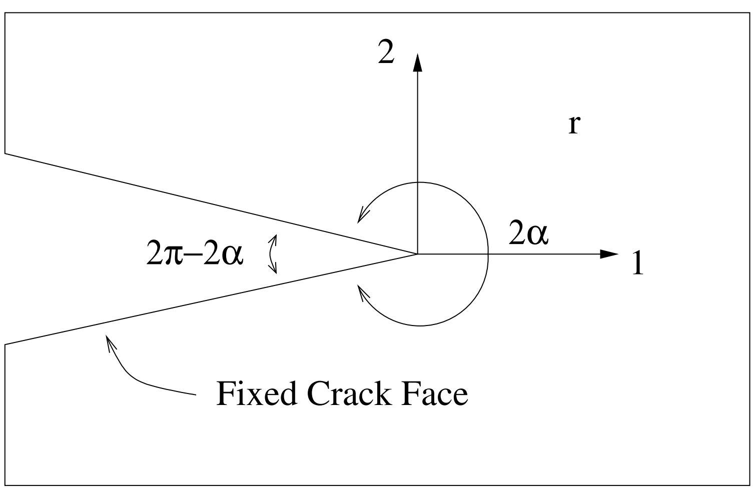
18. Clearly state the required elements of a complete boundary value problem for an arbitrary material; assume small displacement theory.

19. **Thin-Walled Sphere** Consider the thick walled sphere from lecture. Show that before yield, the solution computed for the stress state corresponds to the thin wall sphere solution from elementary mechanics, if one assumes that $(r_o - r_i) << r_i$.

20. In 1952 Max Williams wrote one of the most important papers concerning stress singularities (it has been cited over 900 times): "Stress singularities resulting from various boundary conditions in angular corners of plates in extension", Journal of Applied Mechanics, **19**, 526-528 (1952). A version has been uploaded to bspace.

    1. Read the paper and expand upon the details of how Williams arrived at Eq. [15]. In other words work out the details for the four homogeneous equations to which he refers. Note you do <u>not</u> need to reduce the determinant to Eq. [15] just set up the necessary equations.
    2. Discuss the meaning of Fig. 1 using accurately drawn figures to indicate when one has to and when one does not have to be worried about stress singularities for the three cases in the figure.

21. Consider the anti-plane strain (elastic) stress singularity problem. In the case where the crack faces, located at $\theta = \pm\pi$, are traction-free, one finds that the strength of the stress singularity is $-\frac{1}{2}$; i.e. that the stresses are $O(r^{-\frac{1}{2}})$. Consider now the case where the crack faces are located at $\theta = \pm\alpha$ where $\alpha$ varies between $\pi$ and 0. How does the strength of the stress singularity change as a function of $\alpha$, if we leave the top crack face traction-free BUT restrain the bottom face from moving, $u_3(r, -\alpha) = 0$? For what value of $\alpha$ does the singularity disappear?




22. Consider a linear elastic body $\Omega$ with boundary $\partial\Omega = \partial\Omega_u \cup \partial\Omega_t$, where the displacements are prescribed to be zero on $\partial\Omega_u$ and the tractions on $\partial\Omega_t$. The Hu-Washizu variational

principle assumes independent fields for the displacement, strain, and stress:

$$\min \Pi(\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\tau}) = \int_\Omega \frac{1}{2}\varepsilon_{ij}\mathbb{C}_{ijkl}\varepsilon_{kl} + \sigma_{ij}[\frac{1}{2}(u_{i,j} + u_{j,i}) - \varepsilon_{ij}] - \int_{\partial\Omega_t} u_i \bar{t}_i - \int_\Omega u_i B_i$$

Determine the Euler-Lagrange equations (strong form equations) for this variational principle. Provide a brief description of how one would use this variational principle to solve a problem.

23. Show that the three field Hu-Washizu variational principle represents equilibrium, the elastic constitutive law, and the strain displacement law.

$$\Pi^{HW}(u, \sigma, \varepsilon) = \int_0^L \frac{1}{2}\varepsilon E \varepsilon + \sigma(\frac{du}{dx} - \varepsilon) - bu\, dx - \bar{\sigma}u(L)\,. \tag{74}$$

Assume this is posed for a system that has a fixed displacement at $x = 0$ and a fixed stress $\bar{\sigma}$ at $x = L$.

24. Consider a 1-D system of length L that is restrained from motion at $x = 0$ and has a stress $\sigma(L) = \hat{\sigma}$ applied to the other end. Assume a uniform (constant) body force $b$ and a non-linear constitutive law $\sigma = \alpha\varepsilon^\beta$. Determine the displacement field for the bar, $u(x)$, assuming small deformations.

25. Consider an arbitrary material whose constitutive relations is given as $\varepsilon(\tau)$ – i.e. strain as a known function of stress. By considering the weak form of the strain-displacement relation, $\varepsilon = u_{,1}$, show that our canonical 1-D mechanical boundary value problem can be expressed as:

Find $\tau \in \mathcal{S}_\tau = \{\tau(x) \mid \tau_{,1} = -B \text{ and } \tau(L) = \bar{\tau}\}$ such that

$$\int_0^L \delta\tau\varepsilon(\tau)\, dx - \delta\tau(0)\bar{u} = 0 \qquad \forall \delta\tau \in \mathcal{V}_\tau = \{\delta\tau(x) \mid \delta\tau_{,1} = 0 \text{ and } \delta\tau(L) = 0\}.$$

26. Consider a 1D linear elastic system $\mathcal{R} = [0, L]$ with boundary conditions $u(0) = u(L) = 0$ and body force $b(x) = b_o x$, where $b_o \in \mathbb{R}$.

   1. Find $u(x)$ by solving the strong form problem (Navier's form).
   2. Solve using the weak form with respect to the approximate spaces $\hat{\mathcal{S}} = \{u(x) \mid u(x) = Ax(L - x), \quad A \in \mathbb{R}\}$, $\hat{\mathcal{V}} = \{u(x) \mid u(x) = Bx(L - x), \quad B \in \mathbb{R}\}$
   3. Solve using the principle of minimum potential energy using the approximate solution space
   $$\hat{\mathcal{S}} = \left\{ u(x) \,\middle|\, u(x) = \begin{cases} Ax & x < L/2 \\ A(L - x) & x \geq L/2 \end{cases} \right\}$$
   4. Compare your three solutions by (non-dimensionally) plotting the solutions.

27. Consider an elastic 1-D problem where the domain of interest is $(0, 1)$. At $x = 0$ assume $u(0) = -0.1$ and at $x = 1$ assume $\sigma(1) = 1$. Let $E = 10$ and $b(x) = \exp(x)$. Find the exact solution for the displacement field using the strong form equations. Then generate an approximate solution using the principle of minimum potential energy and

$$\hat{\mathcal{S}} = \left\{ u(x) \mid u(x) = \frac{1}{2}Ax^2 + Bx - 0.1 \right\} \subset \mathcal{S} \; = \{u(x) \mid u(0) = -0.1\} \; . \tag{93}$$

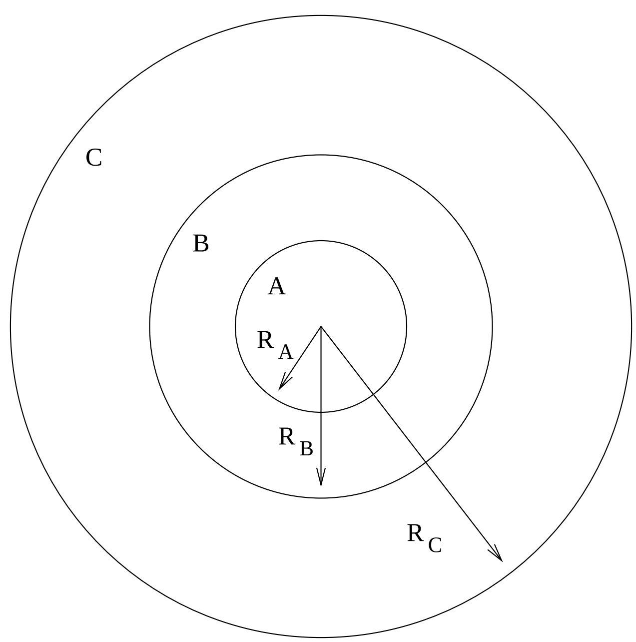
Compare the two solutions by making a plot of the displacements fields.

28. Consider a one-dimensional linear elastic bar subject to displacement boundary conditions $u(0) = u_o$ and $u(L) = u_L$ and a body force $b(x)$. We showed in class that the principle of stationary potential energy is equivalent to the equilibrium statement $\sigma_{,x} + b = 0$. A second principle, known as the principle of stationary complementary potential energy, states that

$$\Pi^c(\sigma) = \int_0^L \frac{1}{2}\sigma^2/E \, dx - [\sigma(L)u(L) - \sigma(0)u(0)]$$

is stationary over the set of stress fields $\mathcal{S} = \{\sigma \mid \sigma_{,x} + b = 0\}$, where $\sigma = E\varepsilon$. By taking the directional derivative of $\Pi^c$ in the direction of a suitably chosen test function $\bar{\sigma}$, show that this principle implies $\varepsilon = u_{,x}$. Hint, first define the space $\mathcal{V}$ in which $\bar{\sigma}$ must lie; note that you need to ensure that sum of an arbitrary element of $\mathcal{S}$ and an arbitrary element of $\mathcal{V}$ is contained in $\mathcal{S}$.

29. Consider a composite body constructed of 3 linear elastic isotropic materials, A, B, and C.



The body is subject to a uniform radial displacement $\boldsymbol{u} = u_C \boldsymbol{e}_r$ at $r = R_C$. Assuming that the displacement field is piecewise continuous of the form

$$\boldsymbol{u} = \begin{cases} u_A r \boldsymbol{e}_r & r < R_A \\ \left[u_A + \frac{u_B - u_A}{R_B - R_A}(r - R_A)\right] \boldsymbol{e}_r & R_A < r < R_B \\ \left[u_B + \frac{u_C - u_B}{R_C - R_B}(r - R_B)\right] \boldsymbol{e}_r & R_B < r < R_C \, , \end{cases}$$

find $u_A$ and $u_B$.

30. Consider a solid isotropic linear-elastic prismatic bar with an arbitrary cross-section whose boundary is described by a known function $g(x_1, x_2) = 0$. A suitable expression for Prandtl's warping function is

$$\psi = g(x_1, x_2) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} f_{mn}(x_1, x_2) \, ,$$

where $f_{mn}(x_1, x_2) = x_1^m x_2^n$. As an approximation, assume all the coefficients $A_{mn}$ are zero except for $A_{00}$, $A_{10}$, and $A_{01}$, and derive a set of 3 linear equations for determining the unknown coefficients.

31. Consider a linear elastic isotropic prismatic bar of arbitrary cross-section clamped at $x = 0$ and subjected to a torque $T$ at $x = L$.

(a) Express the potential energy for the problem in terms of the warping function, load, and twist rate.

(b) Explain how you could use this expression to find an approximate expression for the warping function.

32. **Variational Torsion** Consider the Prandtl stress function $\Psi : \Omega \to \mathbb{R}$. This function satisfies the following PDE

$$\Psi_{,ii} = -2\alpha\mu \qquad \forall \boldsymbol{x} \in \Omega$$

and $\Psi = 0$ for all $\boldsymbol{x} \in \partial\Omega$, $\Omega$ being the cross-section of the bar and $i \in \{1, 2\}$.

1. Set up a weak form statement of this boundary value problem.
2. Show that minimizing the functional

$$I(\Psi) = \int_\Omega \frac{1}{2} \Psi_{,i} \Psi_{,i} - 2\alpha\mu\Psi \, dV$$

is equivalent to your weak form statement of the problem.

33. Consider the following functional:

$$I(\Psi) = \int_A \frac{1}{2}\Psi_{,i}\Psi_{,i} - 2\alpha\mu\Psi \, dA\,, \tag{105}$$

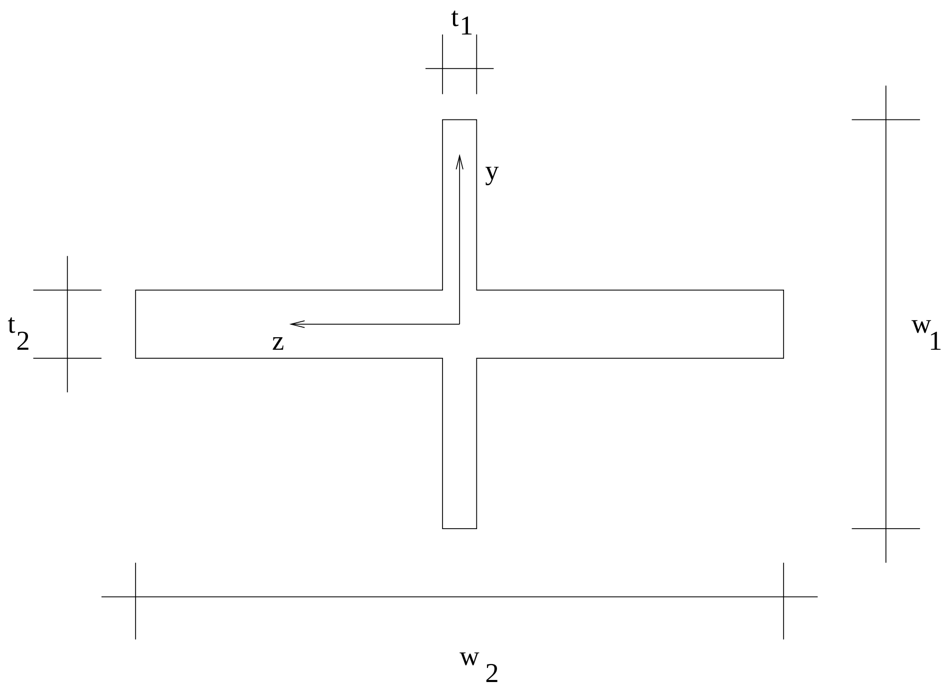
where $A$ represents the cross-section of a prismatic rod, $x_1$ and $x_2$ are the in-plane coordinate directions, and $i \in \{1,2\}$. By defining suitable function spaces (infinite dimensional sets), show that this functional is appropriate for determining Prandtl's stress function for torsion problems; i.e., by making suitable assumptions show that the stationary condition for the given functional is the same as the governing partial differential equation for $\Psi$.

34. Fully develop a weak form problem statement for determining the Prandtl (torsional) stress function over a cross-section $C$ which is governed by $\nabla^2\Psi = -2\mu\alpha$ for points $(y,z) \in C$ subject to the boudary condition $\Psi = 0$ for points $(y,z) \in \partial C$. Note $\nabla^2$ is the 2-D Laplacian $\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Set up an approximate method for solving your weak form problem statement.

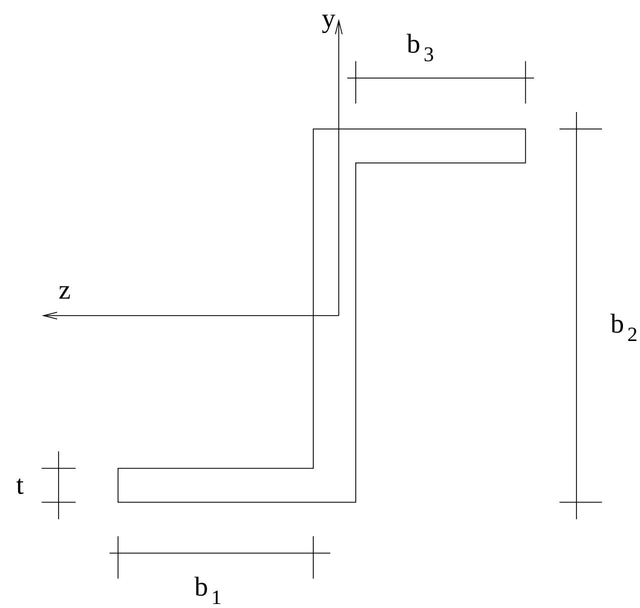
35. **Cruciform torsion** Consider a prismatic beam with the cruicform shape cross-section shown below. Assuming the beam is linear elastic isotropic, you are to determine the torsional stiffness (per unit length of the beam). Do this by assuming an approximate solution of the following form:

$$\Psi = \begin{cases} A(z^2 - t_1^2/4) & (y,z) \in \text{Vertical Section} \\ B(y^2 - t_2^2/4) & (y,z) \in \text{Horizontal Section}\,, \end{cases}$$

where $A$ and $B$ are unknown constants that you should determine by minimizing the potential function that governs this problem (see Problem 32). Note that $\Psi$ does not satisfy all the boundary conditions but it is sufficiently close to give a good result when $t_1$ and $t_2$ are much less than $w_1$ and $w_2$.

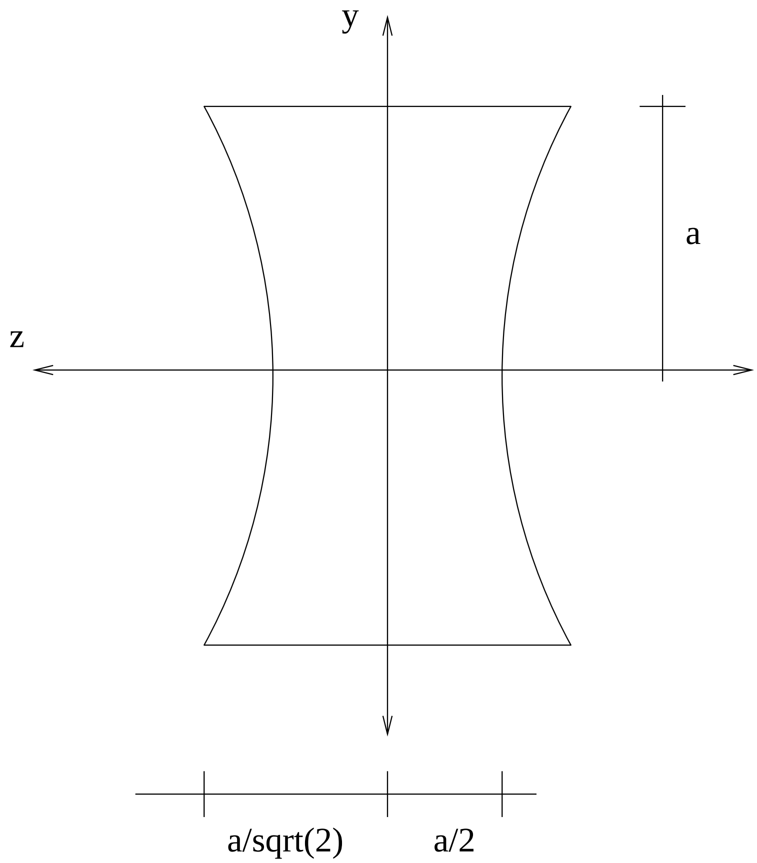



36. **Z-section torsion** Using the same method as in Problem 35 estimate the torsional stiffness per unit beam length for a beam with the cross-sectional shape shown below. Assume $t$ much less than $b_1$, $b_2$, and $b_3$.




37. Shown below is the cross-section of a torsion bar. The vertical sides are generated in the form of hyperbola. The locus of points on the top and bottom is given by $y^2 - a^2 = 0$ and on the sides by $z^2/b^2 - y^2/c^2 - 1 = 0$, where $b = a/2$ and $c = a$. Explain in detail how you would estimate the torsional stiffness per unit length of the bar. An appropriate

answer to this question would include the construction of an explicit trial function basis and a description of the linear equations one would have to solve in order to find the unknown coefficients in the expansion. The elements of the matrix equations should be reasonably well specified but you do not need to perform any integrations.



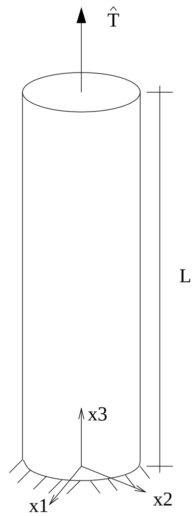

38. **Classical torsion** Consider the isotropic linear elastic system shown below. The classical (strength of materials) governing equations for such a system are given as:

$$\mu J \frac{d^2\phi}{dx_3^2} = 0 \qquad \forall x_3 \in (0, L) \tag{120}$$

$$\phi(0) = 0 \tag{121}$$

$$\mu J \frac{d\phi}{dx_3}(L) = \hat{T}\,, \tag{122}$$

where $\hat{T}$ is a given applied end-torque and $\phi$ is the cross-section rotation.




Derive the classical governing equations (ODE and torque boundary conditions) by minimizing the 3-D potential energy,

$$\Pi(\boldsymbol{u}(\boldsymbol{x})) = \int_{\Omega} \frac{1}{2}\varepsilon_{ij}C_{ijkl}\varepsilon_{kl}\, dV - \int_{A_{end}} t_i u_i \, dA\,, \tag{123}$$

with respect to the approximate set of trial displacement fields

$$\hat{\mathcal{S}} = \{\boldsymbol{u}(\boldsymbol{x}) \mid \boldsymbol{u}(\boldsymbol{x}) = -\phi(x_3)x_2\boldsymbol{e}_1 + \phi(x_3)x_1\boldsymbol{e}_2 + 0\boldsymbol{e}_3 \text{ where } \phi(0) = 0\} \subset \mathcal{S}\,. \tag{124}$$

Note that in this set the approximation "parameters" are not just scalars but rather the function, $\phi(x_3)$.

39. Consider a hollow isotropic linear thermo-elastic sphere. Assume the material has an isotropic thermal expansion coefficient $\alpha$. The inner radius is $a$ and the outer radius is $b$.

    On the inner radius the temperature is prescribed to be $T_a$ and on the outer radius the temperature is prescribed to be $T_b$. The displacements are prescribed to be zero on the inner and outer radii. The hollow sphere is undergoing a chemical reaction which generates a *known* spatially dependent volumetric heating per unit volume of $s(r)$. The temperature field in the hollow sphere obeys an inhomgeneous Laplace equation; i.e. $\nabla^2 T = s(r)$. In the present spherically symmetric case, $\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = s(r)$

    1. Construct a weak problem statement that allows you to determine the temperature field as a function of $r$; i.e. $T(r)$. Explain how you can use this to determine an approximate expression for $T(r)$.
    2. Construct a weak problem statement that allows you to determine the radial deformation field as a function of $r$; i.e. $u_r(r)$. Explain how you can use this to determine an approximate expression for $u_r(r)$.
    3. Describe how you would compute approximations to $\sigma_{rr}(r)$, $\sigma_{\varphi\varphi}(r)$, and $\sigma_{\theta\theta}(r)$ once you have determined your approximation to $u_r$.

[Note: (1) This is a spherically symmetric problem. (2) The differential integration volume in spherical coordinates is $r^2 \sin(\varphi)\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta$]

40. Consider the statement of weak equilibrium for small strain problems. (Assume there are no displacement boundary conditions.)

    1. Explain what this relation implies when one chooses an arbitrary "rigid displacement" for the test function $\bar{\boldsymbol{u}}$.
    2. Explain what this relation implies when one chooses an arbitrary "rigid rotation" for the test function $\bar{\boldsymbol{u}}$.

41. Our mechanical boundary value problem can be expressed in strong form, weak form, and sometimes as a minimization form. It turns out that there are many different kinds of weak forms and minimization forms. The weak form we have looked at also goes by the name of principle of virtual displacements and is just a manipulation of the strong form of the equilibrium equation. Assume an arbitrary material occupying a region $\mathcal{R}$ with displacement boundary conditions $u_i = \bar{u}_i$ on $\partial\mathcal{R}_u$ and traction boundary conditions $\sigma_{ji}n_j = \bar{t}_i$ on $\partial\mathcal{R}_t$, where $\partial\mathcal{R} = \partial\mathcal{R}_u \cup \partial\mathcal{R}_t$ and $\partial\mathcal{R}_t \cap \partial\mathcal{R}_u = \emptyset$.

    Starting from the strong form expression for the strain-displacement relations, show

    $$\int_{\mathcal{R}} \varepsilon_{ij}\delta\sigma_{ij}\,dV = \int_{\partial\mathcal{R}_u} \bar{u}_i\delta t_i\,dA\,, \tag{125}$$

    for all $\delta\boldsymbol{\sigma} \in \{\delta\boldsymbol{\sigma} \mid \delta\boldsymbol{\sigma} = \delta\boldsymbol{\sigma}^T, \nabla\cdot\delta\boldsymbol{\sigma}^T = \boldsymbol{0} \text{ in } \mathcal{R},\ \text{and } \delta\boldsymbol{\sigma}^T\cdot\boldsymbol{n} = \boldsymbol{0} \text{ on } \partial\mathcal{R}_t\}$, where $\delta\boldsymbol{t} = \delta\boldsymbol{\sigma}^T\cdot\boldsymbol{n}$. This is nothing but the principle of virtual forces.