Questions on Elasticity

1. Show that in isotropic linear elasticity the principal axes of stress and strain coincide.

2. In 1971 Tschoegl proposed a free energy function for a carbon-reinforced natural rubber vulcanizate of essentially the following form:

$$\Psi = A(I_1(\boldsymbol{C}) - 3) + B(I_2(\boldsymbol{C}) - 3) + D(I_1(\boldsymbol{C}) - 3)(I_2(\boldsymbol{C}) - 3) + \frac{\lambda}{2}(\ln(J))^2 - 2A\ln(J)$$

   Determine the expression for the 2$^{\text{nd}}$ Piola-Kirchhoff stress tensor.

3. Consider a linear elastic incompressible body ($\nu = 1/2$). What condition must the boundary displacements satisfy?

4. Consider the strain energy of an isotropic linear elastic body

$$E_{\text{body}} = \int_\Omega \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}\varepsilon_{ii}\varepsilon_{jj}\, dV\,.$$

   Argue why the constant μ must necessarily be positive and that $3\lambda+2\mu$ must necessarily be positive. [Hint consider special states of strain and the meaning of E_{body}.] This problem "shows" that the strain energy density should be positive definite.

5. In linear isotropic thermoelasticity we have

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}\,.$$

   Invert this relation to give σ_{ij} as a function of strain and temperature change.

6. The isotropic linear elastic moduli can be expressed as

$$\mathbb{C} = c_1\mathbb{I}^{\text{dev,sym}} + c_2\mathbf{1}\otimes\mathbf{1}\,,$$

   where $\mathbb{I}^{\text{dev,sym}} = \mathbb{I}^{\text{sym}} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}$ is the "symmetric deviatoric projection operator".

   1. Justify the name symmetric deviatoric projection operator for $\mathbb{I}^{\text{dev,sym}}$ by considering its action on an arbitrary 2nd order tensor.
   2. Determine c_1 and c_2 in terms of the bulk modulus and the shear modulus.

7. Motivate the statement: "If the Poisson's ratio is one-half, then the material is incompressible".

8. Starting from $C_{ijkj} = 2\mu\mathbb{I}^{\text{sym}}_{ijkl} + \lambda\delta_{ij}\delta_{kl}$ find an expression for the Young's modulus in terms of μ and λ.

10. **Traction in terms of displacements** For a linear elastic isotropic body, find an expression for the tractions on a surface with normal components n_i in terms of the displacement field and its derivatives as well as the material properties, instead of in terms of the stress tensor.

11. **Poisson's ratio in terms of μ and λ** Starting from $\mathbb{C}_{ijkl} = 2\mu\mathbb{I}^{\text{sym}}_{ijkl} + \lambda\delta_{ij}\delta_{kl}$ find an expression for Poisson's ratio in terms of μ and λ.

12. Consider a body with a homogeneous state of strain: $\boldsymbol{\varepsilon} = (\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{a}\otimes\boldsymbol{b} + \boldsymbol{b}\otimes\boldsymbol{a}) \times 10^{-5}$, where $\boldsymbol{a} = 1\boldsymbol{e}_1 + 1\boldsymbol{e}_2 + 1\boldsymbol{e}_3$ and $\boldsymbol{b} = 2\boldsymbol{e}_1 + 1\boldsymbol{e}_2 + 1\boldsymbol{e}_3$. Assume that the body is isotropic linear elastic so that $\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}$ where the Lamé parameters are $\mu = \lambda = 100$ GPa.

    1. What is the minimum normal stress (in absolute value) and in which direction does it occur?
    2. What is the maximum shear stress (in absolute value) in the body?
    3. Find the deviatoric stress field.
    4. Find the pressure field.

13. Show there are only 3 constant for a linear elastic cubic material.

14. Consider a linear elastic cubic material whose material axes are chosen to line up with three orthonormal vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. Find an indicial expression for the material stiffness $\mathbb{C}_{ijkl}$. [Hint: Start with the isotropic moduli and correct them as needed.]

15. Starting from the linear elastic orthotropic material, show that 90 degree rotational symmetry about the 1, 2, and 3 orthotropic-axes implies there are only 3 independent elastic constants; ie. show a linear elastic cubic material has only three material constants.

16. **Young's Modulus Silicon** The Young's modulus of a material is defined by performing a uni-axial tension test, $\boldsymbol{\sigma} = \sigma_o \boldsymbol{e}_1 \otimes \boldsymbol{e}_1$, and then measuring the resulting axial strain $\varepsilon_o = \boldsymbol{e}_1 \cdot \boldsymbol{\varepsilon}\boldsymbol{e}_1$ in the direction of the load. The Young's modulus is defined, then, as $E = \sigma_o/\varepsilon_o$. For isotropic materials E is a constant value independent of how such a test specimen is cut out of a chunk of material. This however is not the case for anisotropic materials. Plot the variation in Young's Modulus for Silicon as a function of sample orientation with respect to the crystal lattice basis. Note:

    $$\mathbb{C} = 2\gamma\mathbb{I}^{\text{sym}} + \beta\mathbf{1}\otimes\mathbf{1} + (\alpha - \beta - 2\gamma)[\boldsymbol{a}\otimes\boldsymbol{a}\otimes\boldsymbol{a}\otimes\boldsymbol{a} + \boldsymbol{b}\otimes\boldsymbol{b}\otimes\boldsymbol{b}\otimes\boldsymbol{b} + \boldsymbol{c}\otimes\boldsymbol{c}\otimes\boldsymbol{c}\otimes\boldsymbol{c}],$$

    where $\alpha = 166.0$ GPa, $\beta = 64.0$ GPa, and $\gamma = 80.0$ GPa. Also, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are unit vectors aligned with the crystal axes.

17. Consider a sphere of single crystal Silicon with radius 3 cm. You wish to compress the sphere into a sphere of radius of 2.97 cm. What tractions field do you need to apply to the surface of the sphere? Note the elastic constants for Silicon are $\alpha = 166$ GPa, $\beta = 64$ GPa, and $\gamma = 80$ GPa.

18. Consider a single crystal of Silicon in a homogeneous state of strain with cubic elastic constants $\alpha = 166$ GPa, $\lambda = 64$ GPa, and $\mu = 80$ GPa. The lattice vector are known

to be $\boldsymbol{a} = \boldsymbol{e}_2$, $\boldsymbol{b} = (-\boldsymbol{e}_1 + \boldsymbol{e}_3)/\sqrt{2}$, and $\boldsymbol{c} = (\boldsymbol{e}_1 + \boldsymbol{e}_3)/\sqrt{2}$. If the stress state has been measured in the $\{\boldsymbol{e}_i\}_{i=1}^3$ basis to be

$$\boldsymbol{\sigma} \sim \begin{bmatrix} 0 & 10 & 50 \\ 10 & 3 & 9 \\ 50 & 9 & -3 \end{bmatrix}_{123} \quad \text{MPa}\,,$$

Find $\boldsymbol{\varepsilon}$.

19. The (Lamé)-Navier equations of equilibrium for an isotropic material are given as $(\lambda + \mu)u_{k,ki} + \mu u_{i,jj} + f_i = 0$. What do these equations look like for a cubic material, where $\mathbb{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbb{I}_{\text{sym}} + (\alpha - \lambda - 2\mu)[\boldsymbol{a} \otimes \boldsymbol{a} \otimes \boldsymbol{a} \otimes \boldsymbol{a} + \boldsymbol{b} \otimes \boldsymbol{b} \otimes \boldsymbol{b} \otimes \boldsymbol{b} + \boldsymbol{c} \otimes \boldsymbol{c} \otimes \boldsymbol{c} \otimes \boldsymbol{c}]$ with $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ given *unit* vectors along the edges of the crystal axes.

20. Consider a linear elastic orthotropic material where the coordinate basis $\{\boldsymbol{e}_i\}_{i=1}^3$ is aligned with the material axes. The components of the material moduli can be expressed in this basis using Voigt notation as

$$\mathbb{C} \rightarrow \begin{bmatrix} 100 & 50 & 60 & 0 & 0 & 0 \\ 50 & 110 & 70 & 0 & 0 & 0 \\ 60 & 70 & 200 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 35 & 0 \\ 0 & 0 & 0 & 0 & 0 & 45 \end{bmatrix} \quad \text{MPa}\,.$$

Suppose that one now introduces a new basis $\boldsymbol{a}_1 = (\boldsymbol{e}_1 + \boldsymbol{e}_2)/\sqrt{2}$, $\boldsymbol{a}_2 = (\boldsymbol{e}_2 - \boldsymbol{e}_1)/\sqrt{2}$, $\boldsymbol{a}_3 = \boldsymbol{e}_3$. What is $\mathbb{C}_{2233}$ in the $\{\boldsymbol{a}_A\}_{A=1}^3$ basis?

21. Consider a sphere of radius 3 (cm) that is made of linear elastic triclinic Copper Sulfate. A load is applied to the sphere so that it is in a state of pure dilation $\varepsilon_{ij} = \frac{1}{3}\hat{\theta}\delta_{ij}$, where $\hat{\theta}$ is a given constant. Determine the required tractions on the surface of the sphere to create this state of strain. Express your answer in spherical coordinates.

$$\mathbb{C} \rightarrow \begin{bmatrix} 5.709 & 2.062 & 3.164 & -0.426 & -0.042 & -0.221 \\ & 3.577 & 2.34 & -0.281 & -0.012 & -0.058 \\ & & 5.841 & -0.084 & -0.284 & -0.075 \\ & & & 1.65 & -0.185 & 0.119 \\ & \text{sym.} & & & 1.515 & -0.353 \\ & & & & & 1.205 \end{bmatrix} \quad \text{GPa}$$

Comment on what your answer would qualitatively look like if the material were isotropic. Note that the 3-D constitutive relations we developed in class are valid in any orthogonal coordinate system; e.g. $i \in \{x, y, z\}$ or $i \in \{\rho, \theta, \phi\}$.

22. **Copper Sulfate** Consider a cube shaped body of side length l that is made of the linear elastic triclinic material Copper Sulfate. The cube is aligned with the coordiate axes

and centered at the origin. A load is applied to the body so that it is in a state of pure dilation $\varepsilon_{ij} = \frac{\theta}{3}\delta_{ij}$, where θ is a known constant that represents the imposed volumetric strain. Determine the required tractions on the surface of the cube with normal $\boldsymbol{e}_1$. The moduli in the coordinate system aligned with the cube axes are:

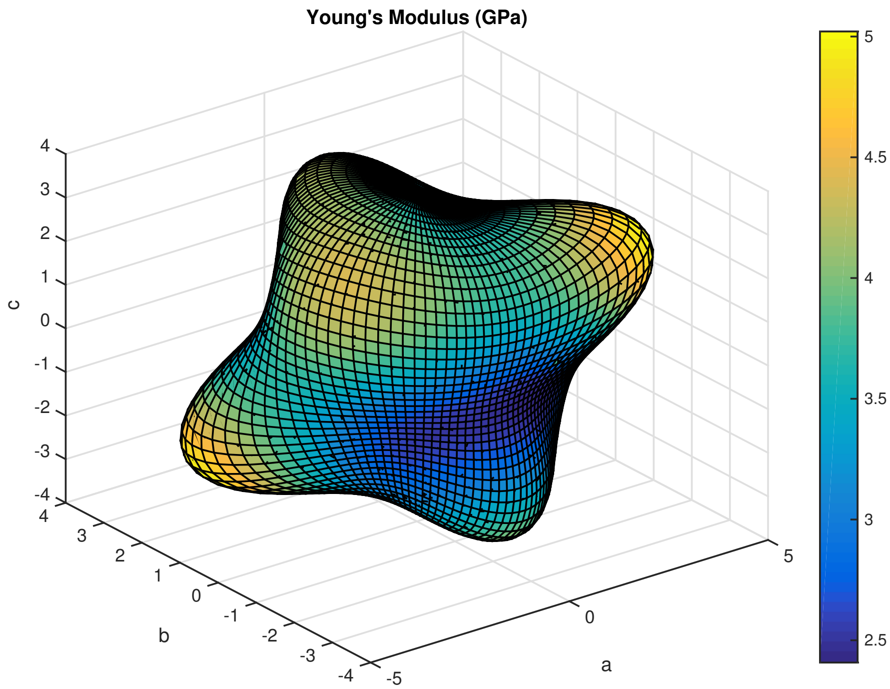
$$\mathbb{C} \rightarrow \begin{bmatrix} 5.709 & 2.062 & 3.164 & -0.426 & -0.042 & -0.221 \\ & 3.577 & 2.340 & -0.281 & -0.012 & -0.058 \\ & & 5.841 & -0.084 & -0.284 & -0.075 \\ & & & 1.650 & -0.185 & 0.119 \\ & \text{sym.} & & & 1.515 & -0.353 \\ & & & & & 1.205 \end{bmatrix} \text{GPa} \tag{29}$$

Comment on what your answer would qualitatively look like if the material were isotropic.

23. The Young's modulus of a material is determined by performing a uni-axial tension test, $\boldsymbol{\sigma} = \sigma_o \boldsymbol{n} \otimes \boldsymbol{n}$, and then measuring the resulting axial strain $\varepsilon_o = \boldsymbol{n} \cdot \boldsymbol{\varepsilon n}$ in the direction of the load. The Young's modulus is defined, then, as $E = \sigma_o/\varepsilon_o$. For isotropic materials E is a constant value independent of how such a test specimen is cut out of a chunk of material. This however is not the case for anisotropic materials. Plot the variation in Young's Modulus for Copper Sulfate as a function of sample orientation with respect to the crystal lattice basis. Note: the moduli for Copper Sulfate in the coordinate system aligned with the crystallographic axes are:

$$\mathbb{C} \rightarrow \begin{bmatrix} 5.709 & 2.062 & 3.164 & -0.426 & -0.042 & -0.221 \\ & 3.577 & 2.340 & -0.281 & -0.012 & -0.058 \\ & & 5.841 & -0.084 & -0.284 & -0.075 \\ & & & 1.650 & -0.185 & 0.119 \\ & \text{sym.} & & & 1.515 & -0.353 \\ & & & & & 1.205 \end{bmatrix} \text{GPa}$$

Your result should match the following figure:



24. Composite materials made with fibers randomly oriented in a single plane typically have transversely isotropic material properties. Such materials have a single plane of reflective symmetry (say with normal $\boldsymbol{e}_3$) and the material is isotropic in that plane (the one with normal $\boldsymbol{e}_3$). Isotropic in the plane, implies that all directions in the plane are elastically equivalent. Starting from the result for orthotropic materials show that such a material has 5 elastic constants.

25. Consider a crystalline tetragonal material with lattice spacings $a = b \neq c$ and lattice angles $\alpha = \beta = \gamma = \frac{\pi}{2}$. Determine the number of unique non-zero elastic constants and any necessary dependencies.

26.

27. Write $A_{ijkl} = 5\delta_{ij}\delta_{kl} + 7\delta_{i1}\delta_{j1}\delta_{k1}\delta_{l1} + 6(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ in Voigt notation.

28.

29. $\mathbb{D} = \frac{1+\nu}{E}\mathbb{I}^{\text{sym}} - \frac{\nu}{E}\mathbf{1}\otimes\mathbf{1}$. What is $\mathbb{D}^{-1}$?

30. **Displacement field equilibrium** Consider a body $\mathcal{R} = [0, L] \times [0, L] \times [0, L]$ whose displacement field is given by

$$\boldsymbol{u} \sim \begin{pmatrix} cx_1^3 + ax_2 \\ bx_3^2 \\ dx_1 \end{pmatrix}$$

Assume that the body has no body forces and that there are no accelerations (static problem). Assuming that the body is linear elastic isotropic, is it in equilibrium? Assume $a, b, c, d \neq 0$.

31. Consider a linear elastic isotropic body $\Omega \subset \mathbb{R}^3$ such that $\Omega = A \times (0, w)$ where $A \subset \mathbb{R}^2$. Thus the body has a uniform thickness w in, say, the x_3 direction The loads on the body consist of body forces of the form $\boldsymbol{b}(\boldsymbol{x}) = b(x_1, x_2)\boldsymbol{e}_3$ and surface tractions on ∂A of the form $\boldsymbol{t}(x_1, x_2) = t(x_1, x_2)\boldsymbol{e}_3$. Assuming that the displacement field in the body is of the form $\boldsymbol{u}(\boldsymbol{x}) = u(x_1, x_2)\boldsymbol{e}_3$, use the principle of minimum potential energy to show that

$$\mu \nabla^2 u + b = 0 \,,$$

where μ is the shear modulus and ∇^2 is the two-dimensional Laplacian.

32. Write the strain energy density of a linear elastic isotropic body in terms of the invariants of the stress tensor.

33. The stress distribution in a linear elastic body is given as

$$\sigma_{xx} = 3x^2 + Axy - 8y^2 \tag{34}$$
$$\sigma_{yy} = 2x^2 + xy + Cy^2 \tag{35}$$
$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \tag{36}$$
$$\sigma_{xy} = -Bx^2 - 6xy - 2y^2 \tag{37}$$
$$\sigma_{xz} = \sigma_{yz} = 0 \,, \tag{38}$$

where ν is Poisson's ratio and A, B, and C are constants.

(a) Determine the values of A, B, and C for this body to be in equilibrium in the absence of body forces.

(b) Does this stress field lead to a single-valued displacement field?

34. For a linear elastic body Ω with boundary Γ, show that

$$\int_\Omega 2W \, d\Omega = \int_\Gamma \boldsymbol{u} \cdot \boldsymbol{t} \, d\Gamma + \int_\Omega \boldsymbol{u} \cdot \boldsymbol{b}_o \, d\Omega \,,$$

where W is the strain energy density for the material.

35. Use Clapeyron's Theorem and the fact that in linear elasticity the strain energy density $W(\cdot)$ is positive definite to prove uniqueness of the linear elastic boundary value problem. In other words, consider a given boundary value problem with imposed body forces, surface tractions, and surface displacements. Show that if there are two solutions to the given boundary value problem, then their difference must necessarily be zero. Note that positive definiteness of the strain energy means $W(\boldsymbol{\varepsilon}) \geq 0$ for all $\boldsymbol{\varepsilon}$ and that the equality holds only for $\boldsymbol{\varepsilon} = 0$.

36. Carefully explain with complete sentences and select equations the connections between the strong, weak, and minimization forms of 3-D linear elasticity.

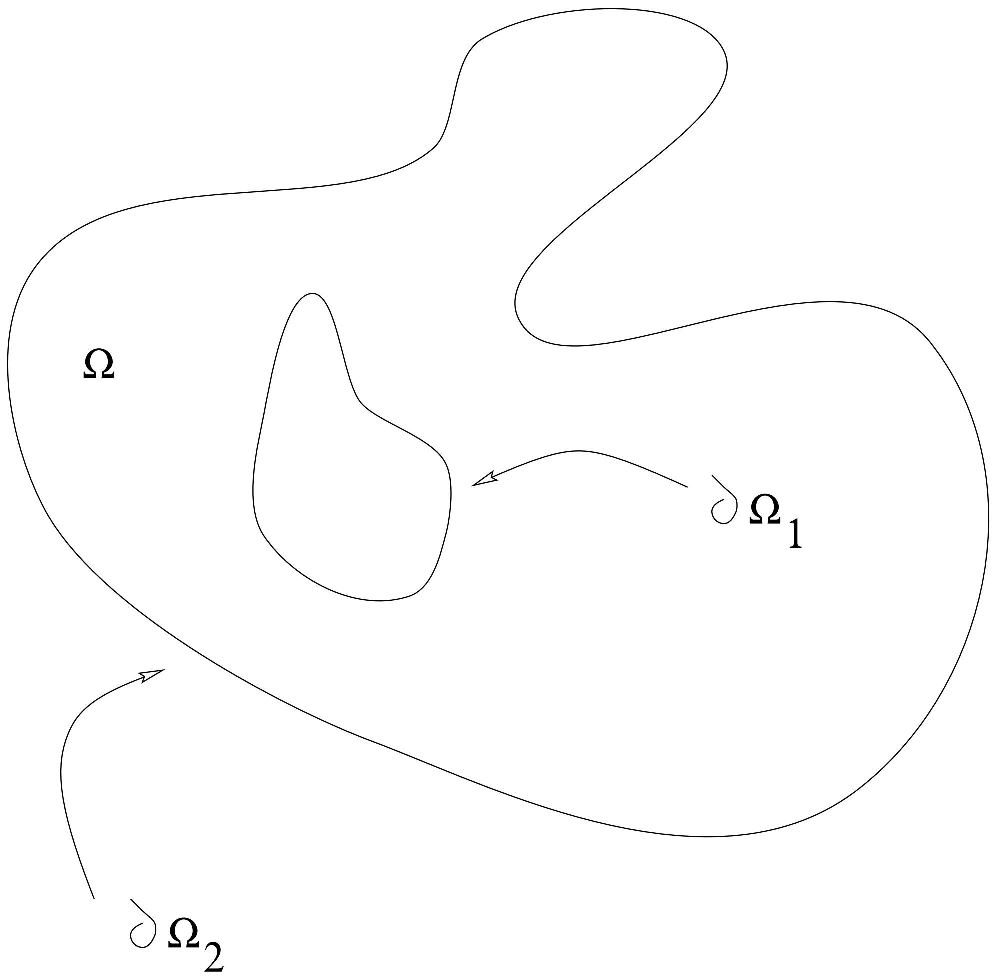
37. Consider the linear elastic body shown. When the applied boundary loads are

$$\bar{\boldsymbol{t}}(\boldsymbol{x}) = -p\boldsymbol{n} \qquad \forall \boldsymbol{x} \in \partial\Omega_2 \tag{48}$$
$$\bar{\boldsymbol{t}}(\boldsymbol{x}) = 0\boldsymbol{n} \qquad \forall \boldsymbol{x} \in \partial\Omega_1 \tag{49}$$

the solution for the stresses is known to be given by $\boldsymbol{\tau}^*(\boldsymbol{x})$, where $\boldsymbol{n}$ is the boundary normal vector. Find the solution for the stress field when the applied loads are given as

$$\bar{\boldsymbol{t}}(\boldsymbol{x}) = 0\boldsymbol{n} \qquad \forall \boldsymbol{x} \in \partial\Omega_2 \tag{50}$$
$$\bar{\boldsymbol{t}}(\boldsymbol{x}) = -p\boldsymbol{n} \qquad \forall \boldsymbol{x} \in \partial\Omega_1 \tag{51}$$

Express your answer in terms of $\boldsymbol{\tau}^*(\boldsymbol{x})$ and p.



38. Consider Eshelby's energy-momentum tensor for elastic bodies with zero body forces

$$P_{ij} = W(\boldsymbol{\varepsilon})\delta_{ij} - \sigma_{li}u_{l,j}$$

where $W(\boldsymbol{\varepsilon})$ is the strain energy density of the material. Show that $P_{ij,i} = 0$.

39. **Maxwell-Betti Relation** Consider a linear elastic body Ω of general symmetry and two different stress-strain fields over the body, $(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\varepsilon}^{(1)})$ and $(\boldsymbol{\sigma}^{(2)}, \boldsymbol{\varepsilon}^{(2)})$. Show that

$$\int_\Omega \boldsymbol{\sigma}^{(1)} : \boldsymbol{\varepsilon}^{(2)} = \int_\Omega \boldsymbol{\sigma}^{(2)} : \boldsymbol{\varepsilon}^{(1)} .$$

40. The potential energy of a linear elastic body Ω under given surface tractions, $\boldsymbol{t}$, and body forces, $\boldsymbol{b}$ is

$$\Pi(\boldsymbol{u}) = \int_\Omega \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}\, d\Omega - \int_\Omega \boldsymbol{b} \cdot \boldsymbol{u}\, d\Omega - \int_{\partial\Omega} \boldsymbol{t} \cdot \boldsymbol{u}\, d\Gamma$$

The actual displacement field $\boldsymbol{u}$ renders Π a minimum. Show that this implies the statement of virtual work:

$$\int_\Omega \boldsymbol{\sigma} : \nabla^s \boldsymbol{\eta}\, d\Omega = \int_\Omega \boldsymbol{b} \cdot \boldsymbol{\eta}\, d\Omega + \int_{\partial\Omega} \boldsymbol{t} \cdot \boldsymbol{\eta}\, d\Gamma$$

for all vector fields $\boldsymbol{\eta}$, where $\nabla^s \boldsymbol{\eta} = \frac{1}{2}(\nabla \boldsymbol{\eta} + (\nabla \boldsymbol{\eta})^T)$. Further show that this implies

$$\begin{aligned} \operatorname{div}[\boldsymbol{\sigma}] + \boldsymbol{b} &= 0 & \qquad \boldsymbol{x} &\in \Omega \\ \boldsymbol{\sigma}\boldsymbol{n} &= \boldsymbol{t} & \qquad \boldsymbol{x} &\in \partial\Omega \end{aligned}$$