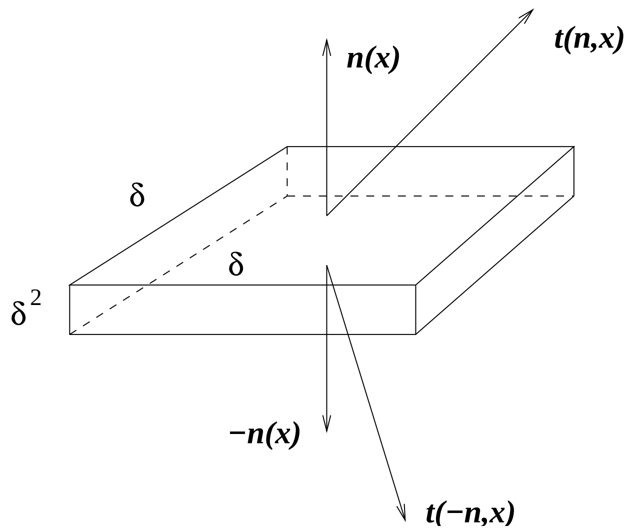
Questions on Balance

1. **Linear momentum balance** Express linear momentum balance in strong form.

2. **Angular momentum balance** Express angular momentum balance in strong form.

3. **"Newton's 3rd Law"** Prove "Newton's 3rd Law" $\boldsymbol{t}(\boldsymbol{n}, \boldsymbol{x}) = -\boldsymbol{t}(-\boldsymbol{n}, \boldsymbol{x})$ by applying global linear momentum balance to a subset $\mathcal{P}_\delta \subset \mathcal{B}$. Here, $\mathcal{P}_\delta$ is a rectangular region centered at $\boldsymbol{x}$, which has dimensions $\delta \times \delta \times \delta^2$, and which has $\boldsymbol{n}$ and $-\boldsymbol{n}$ normal to the $\delta \times \delta$ faces.



4. **Sufficiency in Cauchy's Theorem** Prove sufficiency in Cauchy's Theorem; i.e. given $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, and $\text{div}[\boldsymbol{\sigma}] + \boldsymbol{b}_o = \rho\ddot{\boldsymbol{u}}$ show global linear and angular momentum balance hold for any part of a body.

5. **Couple stresses** Suppose Cauchy's principle is modified to include, in addition to the surface traction $\boldsymbol{t}$, a surface couple (torque) per unit area $\boldsymbol{c}$. Therefore, on a surface we have both a "regular" traction vector and a contact couple vector, $\boldsymbol{c}$.

   Further, suppose that the external loading consists of body forces $\boldsymbol{b}_o$ per unit volume and body torques $\boldsymbol{r}_o$ per unit volume.

   (1) Give an expression for the total force $\boldsymbol{f}$ on a part $\mathcal{P} \subset \mathcal{B}$ of the body. [Hint: Same as before.]

   (2) Give an expression for the total torque (moment) $\boldsymbol{m}$ on a part $\mathcal{P} \subset \mathcal{B}$ of the body. [Hint: Regular expression plus two additional terms.]

   (3) Formulate Newton's Laws of Motion in integral form for a part $\mathcal{P} \subset \mathcal{B}$ of the body. Note, linear momentum $\boldsymbol{l} = \int_{\mathcal{P}} \rho\boldsymbol{v}\, dv$ and angular momentum $\boldsymbol{h} = \int_{\mathcal{P}} \boldsymbol{x} \times \rho\boldsymbol{v}\, dv$, where moments are taken about the fixed origin.

(4) Mimic the tetrahedron argument in Cauchy's Theorem to establish the existance of a couple stress tensor $\boldsymbol{\mu}$ such that

$$\boldsymbol{c} = \boldsymbol{\mu}\boldsymbol{n}$$

(5) Localize the equations of part (3) to determine the partial differential equations for the balance of linear and angular momentum. Note: (a) In the case without couple stresses angular momentum balance reduces to $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$; here you will obtain a partial differential equation. (b) You may employ the result $\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n}$ without proof. (c) Use the convention of the first index of a stress tensor identifying a force/couple direction and the second index identifying the face.

(6) Comment on the symmetry of $\boldsymbol{\sigma}$ in this new setting.

6. **Traction components** Given the following state of stress at a point of interest

$$\boldsymbol{\sigma} \to \begin{bmatrix} 3 & 4 & -8 \\ 4 & -2 & 6 \\ -8 & 6 & 5 \end{bmatrix} \text{ksi} \tag{15}$$

what are the components of the traction vectors on the planes with normals $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$.

7. **Normal traction** For the state of stress in Problem 6, what is the normal stress on the plane with normal $\boldsymbol{n} = (\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)/\sqrt{3}$?

8. **Shear traction** For the state of stress in Problem 6, what is the maximum shear stress on the plane with normal $\boldsymbol{n} = (\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)/\sqrt{3}$?

9. **Cauchy stress from tractions** A square plate $\mathcal{R} = [0, 2] \times [0, 2]$ is deformed such that the deformation map is given by $\boldsymbol{\chi}(X_1, X_2) = (X_1 + \gamma X_2)\boldsymbol{e}_1 + X_2\boldsymbol{e}_2$, where γ is a given constant. At the center of the plate the traction vector on a surface with normal $\boldsymbol{e}_2$ is

$$\boldsymbol{t}(\boldsymbol{e}_2) = a\boldsymbol{e}_1 + b\boldsymbol{e}_2\,,$$

where a and b are known constants. Likewise, at the center of the plate, the traction vector on a surface with normal $\boldsymbol{n} = [1\boldsymbol{e}_1 - \gamma\boldsymbol{e}_2]/\sqrt{1+\gamma^2}$ is

$$\boldsymbol{t}(\boldsymbol{n}) = c\boldsymbol{e}_1 + d\boldsymbol{e}_2\,,$$

where c and d are known constants. Determine the components of the Cauchy stress tensor at the center of the plate in the $\{\boldsymbol{e}_k\}_{k=1}^2$ basis. Are the parameters (a, b, c, d) independent of each other? or do they satisfy some inter-relations?

10. **Universal solution for hydrostatic loading** Show for a solid body (without holes) subjected to zero body forces and to a surface traction $\boldsymbol{t} = c\boldsymbol{n}$, where $c \in \mathbb{R}$ is a constant and $\boldsymbol{n}$ is the surface normal, that the stress field is given by $\boldsymbol{\sigma}(\boldsymbol{x}) = c\mathbf{1}$.

11. **Signorini's theorem** Define the volume average stress in a body Ω with volume V by

$$\bar{\boldsymbol{\sigma}} = \frac{1}{V} \int_{\Omega} \boldsymbol{\sigma} \, d\Omega$$

Show that in the static case, that

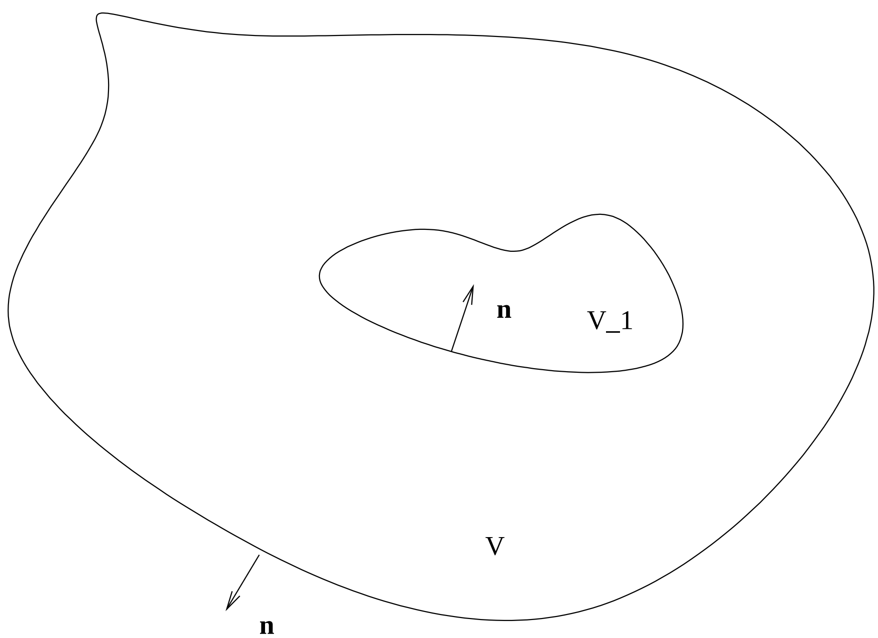
$$\bar{\boldsymbol{\sigma}} = \frac{1}{V} \left[\int_{\partial\Omega} \boldsymbol{t} \otimes \boldsymbol{x} \, d\Gamma + \int_{\Omega} \boldsymbol{b}_o \otimes \boldsymbol{x} \, d\Omega \right]$$

This last result is known as Signorini's Theorem.

12. **Average stress** Consider a body of volume V which contains a single cavity of volume V_1. The surface of the cavity is subject to a traction field $\boldsymbol{t} = p_1 \boldsymbol{n}$, where p_1 is a given constant and $\boldsymbol{n}$ is the outward unit normal vector. The outer surface of the body is subjected to a traction field $\boldsymbol{t} = p_2 \boldsymbol{n}$. Show that the volume average stress in the body is given by

$$\bar{\boldsymbol{\sigma}} = \frac{V_2 p_2 - V_1 p_1}{V} \mathbf{1} \,,$$

where $V_2 = V + V_1$



13. **Stress power** The local stress power is defined as $\mathcal{P} = \sigma_{ij} d_{ij}$ where $\boldsymbol{d} = \mathrm{sym}[\partial \boldsymbol{v} / \partial \boldsymbol{x}]$ is the symmetric velocity gradient. Show that one can express the local stress power as $\mathcal{P} = (1/J) P_{iA} \dot{F}_{iA}$ where $\boldsymbol{P} = J \boldsymbol{\sigma} \boldsymbol{F}^{-T}$ is known as the first Piola-Kirchhoff stress tensor.

14. **Referential equilibrium** For the static case and zero body forces, show that $\sigma_{ij,j} = 0$ implies $P_{iA,A} = 0$.

15. **Differential construction for angular momentum balance** Show symmetry of the Cauchy stress tensor using a classical differential element construction.

16. **Deviatoric projection tensor** If $\sigma_{ij} A_{ijkl} \sigma_{kl} = s_{ij} s_{ij}$ determine A_{ijkl}.

17. **Equilibrium conditions** The stress distribution in a body Ω is given as

$$\begin{aligned} \boldsymbol{\sigma} &= \left(3x_1^2 + Ax_1x_2 - 8x_2^2\right) \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 \\ &+ \left(-Bx_1^2 - 6x_1x_2 - 2x_2^2\right) \left(\boldsymbol{e}_1 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_1\right) \\ &+ \left(2x_1^2 + x_1x_2 + Cx_2^2\right) \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 \,, \end{aligned}$$

where all scalar multipliers and the constants A, B, and C have units of force per unit length[4]. For what values of A, B, and C does this stress distribution represent an equilibrium stress distribution (assume zero body forces and no accelerations).

18. **Integral theorem** Consider a deformable body Ω with boundary $\partial\Omega = \overline{\partial_u\Omega \cup \partial_\tau\Omega}$ where $\partial_u\Omega \cap \partial_\tau\Omega = \emptyset$. Assume $\forall \boldsymbol{x} \in \Omega$ that $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, $\sigma_{ij,j} + b_{oi} = 0$, and $\sigma_{ij} = \sigma_{ji}$; $\forall \boldsymbol{x} \in \partial_u\Omega$ that $u_i = \bar{u}_i$ (given); and $\forall \boldsymbol{x} \in \partial_\tau\Omega$ that $\tau_{ij} n_j = \bar{t}_i$ (given). Show that

$$\int_\Omega \varepsilon_{ij}\sigma_{ij} = \int_{\partial_\tau\Omega} \bar{t}_i u_i + \int_{\partial_u\Omega} \sigma_{ij} n_j \bar{u}_i + \int_\Omega b_{oi} u_i$$

19. **Analysis of stress state**

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \begin{array}{rlrlrl} x_1^2 & \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 & + \; x_2^2 & \boldsymbol{e}_1 \otimes \boldsymbol{e}_2 & - \; 2x_3(x_1 + x_2) & \boldsymbol{e}_1 \otimes \boldsymbol{e}_3 \\ +x_2^2 & \boldsymbol{e}_2 \otimes \boldsymbol{e}_1 & - \; x_2^2 & \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 & + \; 7 & \boldsymbol{e}_2 \otimes \boldsymbol{e}_3 \\ -2x_3(x_1 + x_2) & \boldsymbol{e}_3 \otimes \boldsymbol{e}_1 & + \; 7 & \boldsymbol{e}_3 \otimes \boldsymbol{e}_2 & - \; x_3^2 & \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 \end{array} \tag{41}$$

in a body occuping the domain $\Omega = \{x_i \mid 0 < x_1 < 1\,, \quad 0 < x_2 < 1\,, \quad 0 < x_3 < 4\}$.

    1. What is the total force acting on the surface $x_1 = 1$?
    2. What are the maximum and minimum normal stresses at the center of the body?
    3. What is the maximum shear stress of the center of the body?
    4. If there are no body forces acting in Ω, does $\boldsymbol{\sigma}$ represent an equilibrated stress field?
    5. What is the deviatoric stress at $(1, 1, 0)$? What is the pressure at $(1, 1, 0)$?

20. **Analysis of stress state** Consider a body Ω where the Cauchy stress, $\boldsymbol{\sigma}$, has been measured. The principal stresses are found to be

$$\sigma_1 = 0\,, \quad \sigma_2 = \sqrt{x_1^2 + x_2^2}\,, \quad \sigma_3 = -\sqrt{x_1^2 + x_2^2}\,.$$

and the corresponding principal directions are found to be

$$\begin{aligned}
\boldsymbol{n}_1 &= \frac{x_1}{\sqrt{x_1^2+x_2^2}}\boldsymbol{e}_1 + \frac{x_2}{\sqrt{x_1^2+x_2^2}}\boldsymbol{e}_2 \\
\boldsymbol{n}_2 &= \frac{1}{\sqrt{2}}\left[\frac{x_1}{\sqrt{x_1^2+x_2^2}}\boldsymbol{e}_2 - \frac{x_2}{\sqrt{x_1^2+x_2^2}}\boldsymbol{e}_1 + \boldsymbol{e}_3\right] \\
\boldsymbol{n}_3 &= \frac{1}{\sqrt{2}}\left[\frac{x_1}{\sqrt{x_1^2+x_2^2}}\boldsymbol{e}_2 - \frac{x_2}{\sqrt{x_1^2+x_2^2}}\boldsymbol{e}_1 - \boldsymbol{e}_3\right] .
\end{aligned}$$

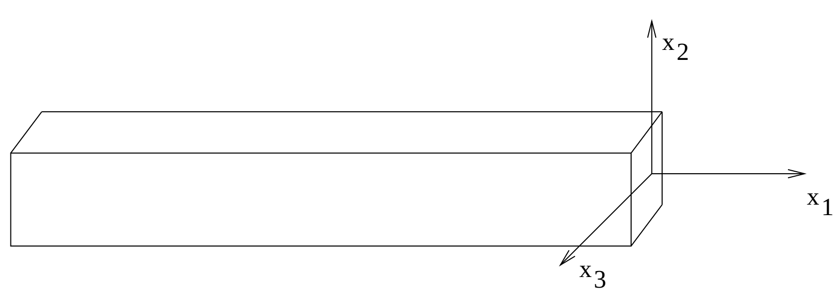
1. Find the components of $\boldsymbol{\sigma}$ in the $\{\boldsymbol{e}_i\}_{i=1}^3$ basis.
2. Find the principal invariants of $\boldsymbol{\sigma}$.
3. Consider any surface with normal $\boldsymbol{e}_3$. What is the traction vector on such a surface?

21. **Uniaxially Loaded Bar** Consider a prismatic bar with axis along the 1-direction in the reference configuration. Assume that the bar is in a homogeneous state of uniaxial stress (along its axis) but that its current orientation is along the $(1,1,1)$-direction. What is the Cauchy stress tensor? What is the 1st Piola-Kirchhoff stress tensor? You may assume that

$$\boldsymbol{F} = \lambda \boldsymbol{v}_1\boldsymbol{e}_1 + \boldsymbol{v}_2\boldsymbol{e}_2 + \boldsymbol{v}_3\boldsymbol{e}_3\,,$$

where $\boldsymbol{v}_1 = (1/\sqrt{3})(\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3)$, $\boldsymbol{v}_2 = (1/\sqrt{2})(\boldsymbol{e}_1-\boldsymbol{e}_2)$, and $\boldsymbol{v}_3 = (1/\sqrt{6})(\boldsymbol{e}_1+\boldsymbol{e}_2-2\boldsymbol{e}_3)$. Assume that λ is a given constant.

22. **Equilibrium equations for rods as averages of the 3D equilibrium equations** Consider a prismatic beam made from an arbitrary material which is subjected to tractions and body forces in the x_1 and x_2 directions. The cross-sectional area is $A = b\cdot h$, where b is the width in the 3-direction and h is the depth in the 2-direction.



Assume the following stresses are zero: $\sigma_{33} = \sigma_{23} = \sigma_{13} = 0$. Allow for general surface tractions within these assumptions. Assume there is no body force in the 3-direction but do **not** assume the other two body force components are are zero. Assume there are no dependencies upon x_3.

1. Define the axial force resultant on a cross-section as

$$P = \int_A \sigma_{11}\, dA \tag{49}$$

and the shear force resultant on the cross-section as

$$V = \int_A \sigma_{12}\, dA\,. \tag{50}$$

By integrating the equilibrium equations for the stresses over the cross-section, show that

$$P_{,1} + p = 0 \tag{51}$$
$$V_{,1} + q = 0\,. \tag{52}$$

Provide suitable definitions for the p and q and argue why they are appropriate.

2. Define the moment resultant on the cross-section as

$$M = -\int_A x_2 \sigma_{11}\, dA\,. \tag{53}$$

Consider the first moment of the equilibrium equations for the stresses and show

$$M_{,1} + V + m = 0 \tag{54}$$

Provide a suitable definition for m and argue why it is appropriate. [The first moment of any quantity f, in this context, is simply $\int_A x_2 f\, dA$.]

23. **Virtual infinitesimal rotations** Consider the weak equilibrium equations for a body subject to arbitrary body forces and surface tractions. Assume there are no displacement boundary conditions; i.e. $\partial\mathcal{R}^u = \emptyset$. Now, consider a test function (virtual displacement) of the form

$$\delta u_i = e_{ijk}\delta\breve{\omega}_j x_k$$

(an infinitesimal rotation). Note, $\delta\breve{\omega}_j$ is arbitrary and constant (not a function of position). If you use this test function in the weak equilibrium (virtual work) equations, what important principle of mechanics do you recover?

24. **Local balance of energy** Consider a body $\mathcal{R}$ that is being deformed in time. The *global* statement of energy (or power) balance/conservation is *given* as

$$\frac{d}{dt}\int_{\mathcal{R}'_t} \frac{1}{2}\rho v_i v_i\, dV + \frac{d}{dt}\int_{\mathcal{R}'_t} \rho e\, dV = \int_{\mathcal{R}'_t} b_i v_i\, dV + \int_{\partial\mathcal{R}'_t} t_i v_i\, dA\,, \qquad \forall \mathcal{R}'_t \subset \mathcal{R}_t$$

where new here is e, the internal energy of the body per unit mass. The first term on the left-hand side is the time rate of change of the kinetic energy; the second term is the time rate of change of the internal energy. The first integral on the right-hand side is the power of the body forces (given per unit volume) and the second integral is the power of the surface tractions. Show that the *local* form of this balance law is given by

$$\rho \dot{e} = d_{ij}\sigma_{ij} \, ,$$

where d_{ij}, the rate of deformation tensor, is given by $d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$. Note ρ is the current/spatial density and v_i is the velocity field.